Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 3, pp. 325-339

DOI: 10.4208/aamm.10-m11102
June 2012

A Novel Perturbative Iteration Algorithm for Effective and Efficient Solution of Frequency-Dependent Eigenvalue Problems

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Received 28 May 2011; Accepted (in revised version) 5 November

Available online 30 April 2012

Abstract. Many engineering structures exhibit frequency dependent characteristics and analyses of these structures lead to frequency dependent eigenvalue problems. This paper presents a novel perturbative iteration (PI) algorithm which can be used to effectively and efficiently solve frequency dependent eigenvalue problems of general frequency dependent systems. Mathematical formulations of the proposed method are developed and based on these formulations, a computer algorithm is devised. Extensive numerical case examples are given to demonstrate the practicality of the proposed method. When all modes are included, the method is exact and when only a subset of modes are used, very accurate results are obtained.

AMS subject classifications: 34L15, 34L16, 35P15, 35P30

Key words: Frequency dependent eigenvalue problem, structural vibration, computer algorithm, damped structural vibration, perturbative iteration, eigenvalue and eigenvector sensitivity.

1 Introduction

Frequency dependent systems are those whose system characteristics change with externally applied stimulating frequency. Many natural practical systems are found to be substantially frequency dependent. These include mechanical engine mounts [1], rotor systems [2], viscoelastic and composite structures [3, 4], human pelvis systems [5], electrical inductances [6], magnetic modulation systems [7] etc. When such systems are to be analyzed and identified to establish their dynamic characteristics, frequency

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dependent eigenvalue problems are often encountered which need to be solved. A frequency depenedent eigenvalue problem can be defined as

$$\mathbf{A}(\lambda)\boldsymbol{\varphi} = \lambda\mathbf{B}(\lambda)\boldsymbol{\varphi}, \tag{1.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are system matrices which can be complex in general but symmetric and which are functions of eigenvalue (frequency) λ while its conventional counterpart is written as

$$\mathbf{A}\boldsymbol{\varphi} = \lambda\mathbf{B}\boldsymbol{\varphi}. \tag{1.2}$$

Eigenvalue problem is a major branch of applied mathematics and has found wide range of applications [8]. Clasical eigenvalue problems are involved in many engineering disciplines and development of effective and efficient eigensolution methods has remained to be very active during the last few decades. To establish a complete set of eigenvalues and eigenvectors, various algorithms have been developed such as Jocobi method [9], QR method [10] and the more general QZ method [11]. While in the case where only few of the eigenvalues and eigenvectors are of interest, inverse iteration method [12] has been developed and can be effectively employed.

For large practical engineering systems, Lanczos method [13] was probably the first method which was successfully applied to solve large scale eigenvalue problems. Lanczos introduced a recursive algorithm based on a Krylov sequence to determine a subset of eigensolutions. Based on similar concept, Arnoldi [14] presented an effective method for large eigenvalue problems. Based on the formulation of an orthogonal set of vectors, an effective algorithm called Ritz method was developed as discussed in [15]. Alongside with these methods, subspace iteration technique, which has been proven to be very efficient and effective, especially for large practical systems, was introduced [16–18]. Nevertheless, these methods were initially developed for undamped systems, when system damping is considered and hence the eigenvalue problem becomes complex in nature, these methods need to be further extended. Complex Lanczos method was discussed intensely in engineering science in 1980's [19,20]. Arnoldi's method was further developed by Saad [21] to tackle eigenvalue problems of damped systems. A generalized Ritz algorithm for quadratic eigenproblems was presented by Zheng et al. [22].

For conventional eigenvalue problems, extensive research has been conducted and many effective and efficient methods have been developed to date to solve them as the literature review above has indicated. However, frequency dependent eigenvalue problem has not been properly investigated despite its practical importance and there has yet to exist an effective eigensolution method to address this type of eigenvalue problems. This paper seeks to develop a novel perturbative iteration (PI) algorithm which can be used to effectively and efficiently solve frequency dependent eigenvalue problems of general frequency dependent systems. Mathematical formulations of the proposed method are developed and based on these formulations, a numerical algorithm is devised. Extensive numerical case examples are given to demonstrate the

practicality of the proposed method. In the case where all modes are included, the method is exact and the iteration always converges to exact solution. In the case where only a subset of modes are used, very accurate results are obtained. Tremendous computational efficiency is manifested through these case examples.

2 Formulation of problem and conventional solution

Many physical systems involve frequency dependent system parameters, leading to frequency depenedent eigenvalue problems described by Eq. (1.1), which need to be solved to determine their dynamic characteristics. One typical such system is a structure with viscoelastic damping material. Viscoelastic materials are widely and increasingly used to increase structural damping for vibration control applications. Both Young's modulus $E = E(\omega)$ and loss factor $\eta = \eta(\omega)$ of a viscoelastic material are functions of frequency. Such frequency dependency can be seen from the reduced frequency monogram of 3M vicoelastic material ISD 112 of Fig. 1. Structural models with such viscoelastic damping layers will be used as examples to validate the proposed methods in later discussions.

Once the frequency dependent eigenvalue problem is formulated whose mathematical formula is repeated here

$$\mathbf{A}(\lambda)\boldsymbol{\varphi} = \lambda\mathbf{B}(\lambda)\boldsymbol{\varphi}. \tag{2.1}$$

What one seeks to do is to find a suitable method to solve all the eigenvalues and corresponding eigenvectors of interest which satisfy Eq. (2.1). Though there has not been any discussion in the open literature regarding how such problem can be solved, one simple and straightforward method will be to apply conventional eigensolution method in an iterative manner. To compute the eigenvalue and the associated eigenvector of rth mode, following algorithm can be used

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1. Initialize: $\lambda = \lambda_0$, $\mathbf{A} = \mathbf{A}(\lambda_0)$ and $\mathbf{B} = \mathbf{B}(\lambda_0)$;
2. For $k = 1, 2, 3, \cdots$,
3. Solve $\mathbf{A}\boldsymbol{\varphi} = \lambda\mathbf{B}\boldsymbol{\varphi} \Rightarrow \lambda_r^{(k)}$, $\boldsymbol{\varphi}_r^{(k)}$;
4. If $|\lambda - \lambda_r^{(k)}|/|\lambda_r^{(k)}| \leq \varepsilon \Rightarrow$ Stop;
5. Update $\lambda = \lambda_r^{(k)}$, $\mathbf{A} = \mathbf{A}(\lambda)$ and $\mathbf{B} = \mathbf{B}(\lambda)$.

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where λ_0 is the initial guess of the eigenvalue of rth mode of interest and ε is the required computational accuracy. As the iteration continues, the estimated $\lambda_r^{(k)}$ and $\boldsymbol{\varphi}_r^{(k)}$ will converge to their exact solutions λ_r and $\boldsymbol{\varphi}_r$. For different mode, this iteration process will have to be repeated *ab* initio.

During the iteration, each eigensolution required is a conventional eigenvalue problem since both $\mathbf{A}$ and $\mathbf{B}$ are constant. However, in order solve just one eigenvalue

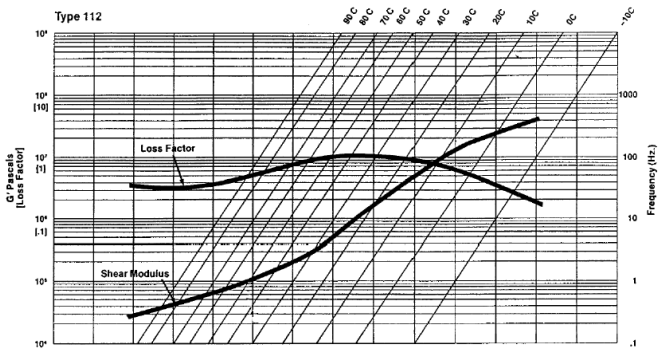


Figure 1: Reduced frequency nomogram for ISD-112.

and its corresponding eigenvector, repeated full eigensolutions are required until required accuracy is met. Such an approach is theoretically feasible but practically prohibitive since the computational cost involved is tremendous even though one might argue that some form of optimization can be done to minimize the computational burden to some extent.

3 Perturbative iteration (Pi) algorithm

The proposed Perturbative Iteration (PI) algorithm seeks to minimize the computational cost involved by performing the costly eigensolution once and only on a companion reference system formed at some pre-selected frequency value $\lambda = \lambda_0$. Subsequently, the eigevalues and eigenvectors of this companion reference system are used together with the perturbation matrices which are the difference between the actual system matrices and those of the companion reference system to derive the true eigenvalues and eigenvectors of interest of the frequency dependent system.

Let the companion reference system be defined as the system corresponding to a known pre-selected frequency $\lambda = \lambda_0$

$$\mathbf{A}_0 = \mathbf{A}(\lambda_0), \qquad \mathbf{B}_0 = \mathbf{B}(\lambda_0). \tag{3.1}$$

The eigenvalue problem of this companion system thus becomes

$$\mathbf{A}_0\overline{\boldsymbol{\varphi}} = \overline{\lambda}\mathbf{B}_0\overline{\boldsymbol{\varphi}}. \tag{3.2}$$

Let us assume here all the eigenvalues $\overline{\lambda}_r$, $r = 1, 2, \cdots, N$ and eigenvectors $\overline{\boldsymbol{\varphi}}_r$, $r = 1, 2, \cdots, N$ of (3.2) are solved using conventional eigensolution techniques mentioned and the effect of incomplete modes on the accuracy of the proposed algorithm will be discussed later. Define further the perturbation matrices as the difference between the actual system matrices and those of the companion reference system as

$$\Delta\mathbf{A}(\lambda) = \mathbf{A}(\lambda) - \mathbf{A}_0, \qquad \Delta\mathbf{B}(\lambda) = \mathbf{B}(\lambda) - \mathbf{B}_0, \tag{3.3}$$

where $\Delta\mathbf{A}(\lambda)$ and $\Delta\mathbf{B}(\lambda)$ are known functions of λ. With these definitions, the eigenvalue problem of the frequency dependent system becomes

$$[\mathbf{A}_0 + \Delta\mathbf{A}(\lambda)]\boldsymbol{\varphi} = \lambda[\mathbf{B}_0 + \Delta\mathbf{B}(\lambda)]\boldsymbol{\varphi}. \tag{3.4}$$

After some re-arrangement, (3.4) can be further written as

$$[\mathbf{A}_0 - \lambda\mathbf{B}_0]\boldsymbol{\varphi} = [-\Delta\mathbf{A}(\lambda) + \lambda\Delta\mathbf{B}(\lambda)]\boldsymbol{\varphi}. \tag{3.5}$$

Upon solving for $\boldsymbol{\varphi}$, one has

$$\boldsymbol{\varphi} = [\mathbf{A}_0 - \lambda\mathbf{B}_0]^{-1}[-\Delta\mathbf{A}(\lambda) + \lambda\Delta\mathbf{B}(\lambda)]\boldsymbol{\varphi}. \tag{3.6}$$

From spectral decomposition, the inverse matrix $[\mathbf{A}_0 - \lambda\mathbf{B}_0]^{-1}$ can be computed using the known eigenvalues and eigenvectors of the companion reference system as

$$[\mathbf{A}_0 - \lambda\mathbf{B}_0]^{-1} = \sum_{i=1}^{N} \frac{\overline{\boldsymbol{\varphi}}_i\overline{\boldsymbol{\varphi}}_i^T}{\overline{\lambda}_i - \lambda}. \tag{3.7}$$

Upon substitution of (3.7), (3.6) becomes

$$\boldsymbol{\varphi} = \sum_{i=1}^{N} \frac{\overline{\boldsymbol{\varphi}}_i\overline{\boldsymbol{\varphi}}_i^T}{\overline{\lambda}_i - \lambda}[-\Delta\mathbf{A}(\lambda) + \lambda\Delta\mathbf{B}(\lambda)]\boldsymbol{\varphi}. \tag{3.8}$$

Eq. (3.8) is a general relationship in which N is the order of the system, for rth mode of interest, one has

$$\boldsymbol{\varphi}_r = \sum_{i=1}^{N} \frac{\overline{\boldsymbol{\varphi}}_i\overline{\boldsymbol{\varphi}}_i^T}{\overline{\lambda}_i - \lambda_r}[-\Delta\mathbf{A}(\lambda_r) + \lambda_r\Delta\mathbf{B}(\lambda_r)]\boldsymbol{\varphi}_r. \tag{3.9}$$

To avoid numerical difficulty when $\overline{\lambda}_r$ is too close to λ_r, especially at the starting point of the iteration process, as will be shown later, some modification to (3.9) becomes necessary. Re-write (3.9) to become

$$\begin{aligned}\boldsymbol{\varphi}_r = &\sum_{i=1;i\neq r}^{N} \frac{\overline{\boldsymbol{\varphi}}_i\overline{\boldsymbol{\varphi}}_i^T}{\overline{\lambda}_i - \lambda_r}[-\Delta\mathbf{A}(\lambda_r) + \lambda_r\Delta\mathbf{B}(\lambda_r)]\boldsymbol{\varphi}_r \\ &+ \frac{\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T}{\overline{\lambda}_r - \lambda_r}[-\Delta\mathbf{A}(\lambda_r) + \lambda_r\Delta\mathbf{B}(\lambda_r)]\boldsymbol{\varphi}_r.\end{aligned} \tag{3.10}$$

The second term on the right hand side of (3.9) can be further simplified, recall (3.5) and in the case of rth mode, (3.5) can be written as

$$[\mathbf{A}_0 - \lambda_r\mathbf{B}_0]\boldsymbol{\varphi}_r = [-\Delta\mathbf{A}(\lambda_r) + \lambda_r\Delta\mathbf{B}(\lambda_r)]\boldsymbol{\varphi}_r. \tag{3.11}$$

Using the relationship of (3.11), the second term on the right hand side of (3.9) can be written as

$$\begin{aligned}
&\frac{\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T}{\overline{\lambda}_r-\lambda_r}[-\Delta\mathbf{A}(\lambda_r)+\lambda_r\Delta\mathbf{B}(\lambda_r)]\boldsymbol{\varphi}_r=\frac{\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T}{\overline{\lambda}_r-\lambda_r}[\mathbf{A}_0-\lambda_r\mathbf{B}_0]\boldsymbol{\varphi}_r\\
&=\frac{\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T}{\overline{\lambda}_r-\lambda_r}[\mathbf{A}_0-\overline{\lambda}_r\mathbf{B}_0+\overline{\lambda}_r\mathbf{B}_0-\lambda_r\mathbf{B}_0]\boldsymbol{\varphi}_r\\
&=\frac{\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T}{\overline{\lambda}_r-\lambda_r}[\mathbf{A}_0-\overline{\lambda}_r\mathbf{B}_0]\boldsymbol{\varphi}_r+\frac{\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T}{\overline{\lambda}_r-\lambda_r}(\overline{\lambda}_r-\lambda_r)\mathbf{B}_0\boldsymbol{\varphi}_r. \qquad (3.12)
\end{aligned}$$

From the definition of eigenvalue problem of the companion reference system of (3.2), one can conclude that

$$\frac{\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T}{\overline{\lambda}_r-\lambda_r}[\mathbf{A}_0-\overline{\lambda}_r\mathbf{B}_0]\boldsymbol{\varphi}_r=0\quad(\because\overline{\boldsymbol{\varphi}}_r^T[\mathbf{A}_0-\overline{\lambda}_r\mathbf{B}_0]=0). \qquad (3.13)$$

Hence, (3.12) can be re-written as

$$\frac{\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T}{\overline{\lambda}_r-\lambda_r}[-\Delta\mathbf{A}(\lambda_r)+\lambda_r\Delta\mathbf{B}(\lambda_r)]\boldsymbol{\varphi}_r=\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T\mathbf{B}_0\boldsymbol{\varphi}_r. \qquad (3.14)$$

Substituting (3.14) into (3.10), one has

$$\boldsymbol{\varphi}_r=\sum_{i=1;i\neq r}^{N}\frac{\overline{\boldsymbol{\varphi}}_i\overline{\boldsymbol{\varphi}}_i^T}{\overline{\lambda}_i-\lambda_r}[-\Delta\mathbf{A}(\lambda_r)+\lambda_r\Delta\mathbf{B}(\lambda_r)]\boldsymbol{\varphi}_r+\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T\mathbf{B}_0\boldsymbol{\varphi}_r. \qquad (3.15)$$

In Eq. (3.15), λ_r and $\boldsymbol{\varphi}_r$ are the eigenvalue and its associated eigenvector which need to be solved while other terms involved are all known. It will be shown that a simple iterative algorithm which is called Perturbative Iteration (PI) can be developed based on (3.15) as discussed below

$$\boldsymbol{\varphi}_r^{(k+1)}=\sum_{i=1;i\neq r}^{N}\frac{\overline{\boldsymbol{\varphi}}_i\overline{\boldsymbol{\varphi}}_i^T}{\overline{\lambda}_i-\lambda_r^{(k)}}[-\Delta\mathbf{A}(\lambda_r^{(k)})+\lambda_r^{(k)}\Delta\mathbf{B}(\lambda_r^{(k)})]\boldsymbol{\varphi}_r^{(k)}+\overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T\mathbf{B}_0\boldsymbol{\varphi}_r^{(k)}, \qquad (3.16a)$$

$$\lambda_r^{(k+1)}=\frac{\{\boldsymbol{\varphi}_r^{(k+1)}\}^T[\mathbf{A}_0+\Delta\mathbf{A}(\lambda_r^{(k)})]\{\boldsymbol{\varphi}_r^{(k+1)}\}}{\{\boldsymbol{\varphi}_r^{(k+1)}\}^T[\mathbf{B}_0+\Delta\mathbf{B}(\lambda_r^{(k)})]\{\boldsymbol{\varphi}_r^{(k+1)}\}}. \qquad (3.16b)$$

Normalize $\boldsymbol{\varphi}_r^{(k+1)}$ with respect to $\mathbf{B}$ (mass normalization in structural dynamics)

$$\boldsymbol{\varphi}_r^{(k+1)}=\frac{\boldsymbol{\varphi}_r^{(k+1)}}{\sqrt{\{\boldsymbol{\varphi}_r^{(k+1)}\}^T[\mathbf{B}_0+\Delta\mathbf{B}(\lambda_r^{(k)})]\{\boldsymbol{\varphi}_r^{(k+1)}\}}}. \qquad (3.17)$$

The complete algorithm can be summarized as:

1. Formulating $\mathbf{A}(\lambda)$, $\mathbf{B}(\lambda)$ and selecting suitable value $\lambda=\lambda_0$;

2. Solving companion system $\mathbf{A}_0\overline{\boldsymbol{\varphi}} = \lambda\mathbf{B}_0\overline{\boldsymbol{\varphi}} \Rightarrow \overline{\lambda}_r$ and $\overline{\boldsymbol{\varphi}}_r$;
3. Initializing $\lambda_r^{(0)} = \overline{\lambda}_r$ and $\boldsymbol{\varphi}_r^{(0)} = \overline{\boldsymbol{\varphi}}_r$;
4. For $k = 1, 2, 3, \cdots$;
5. Computing

$$\boldsymbol{\varphi}_r^{(k+1)} = \sum_{i=1; i\neq r}^{N} \frac{\overline{\boldsymbol{\varphi}}_i\overline{\boldsymbol{\varphi}}_i^T}{\overline{\lambda}_i - \lambda_r^{(k)}}[-\Delta\mathbf{A}(\lambda_r^{(k)}) + \lambda_r^{(k)}\Delta\mathbf{B}(\lambda_r^{(k)})]\boldsymbol{\varphi}_r^{(k)} + \overline{\boldsymbol{\varphi}}_r\overline{\boldsymbol{\varphi}}_r^T\mathbf{B}_0\boldsymbol{\varphi}_r^{(k)};$$

6. Computing

$$\lambda_r^{(k+1)} = \frac{\{\boldsymbol{\varphi}_r^{(k+1)}\}^T[\mathbf{A}_0 + \Delta\mathbf{A}(\lambda_r^{(k)})]\{\boldsymbol{\varphi}_r^{(k+1)}\}}{\{\boldsymbol{\varphi}_r^{(k+1)}\}^T[\mathbf{B}_0 + \Delta\mathbf{B}(\lambda_r^{(k)})]\{\boldsymbol{\varphi}_r^{(k+1)}\}};$$

7. Normalizing

$$\boldsymbol{\varphi}_r^{(k+1)} = \frac{\boldsymbol{\varphi}_r^{(k+1)}}{\sqrt{\{\boldsymbol{\varphi}_r^{(k+1)}\}^T[\mathbf{B}_0 + \Delta\mathbf{B}(\lambda_r^{(k)})]\{\boldsymbol{\varphi}_r^{(k+1)}\}}};$$

8. If $|\lambda_r^{(k+1)} - \lambda_r^{(k)}|/|\lambda_r^{(k+1)}| \leq \varepsilon \Rightarrow$ Stop.

The converged eigenvalue $\lambda_r^{(k+1)}$ and eigenvector $\boldsymbol{\varphi}_r^{(k+1)}$, will approach to their true counterparts λ_r and $\boldsymbol{\varphi}_r$. Other modes of interest can be similarly computed. As compared with the conventional iterative full eigensolution based method which is computationally prohibitive, the proposed PI algorithm is computationally extremely efficient since it only requires simple matrix-vector multiplications and as a result, only a tiny fraction of additional computational effort is required for each mode of interest after the initial eigensolution of the companion reference system, as will be sown later in numerical case examples. Throughout the PI process, there is no matrix multiplication or matrix inverse operations involved and the sparsity of the system matrices, if any, is preserved and can be taken into consideration to further improve the efficiency of the method.

4 Application case examples

To assess the computational performance of the proposed PI algorithm, an undamped vibration problem of a mass-spring model shown in Fig. 2 was first examined.

The value for the spring elements k_3 shown in Fig. 2 was assumed to be frequency dependent and it is assumed to be modeled as

$$k_3 = 3000000 + 200000\lambda^{0.0875}\mathrm{N}/m. \tag{4.1}$$

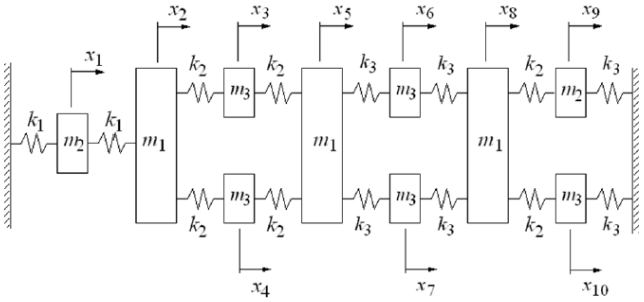


Figure 2: A mass-spring vibration model used as numerical example with $m_1 = 1.0$kg, $m_2 = 0.2$kg, $m_3 = 0.1$kg, $k_1 = 1.0 \times 10^6$N/m, $k_2 = 2.0 \times 10^6$N/m.

Upon deriving the equations of motion and putting them in matrix form, a 10×10 mass matrix $\mathbf{B}$ which is diagonal and a stiffness matrix $\mathbf{A}$ of the same dimension, which is frequency dependent, can be established. Let the companion reference system be

$$\mathbf{B}_0 = \mathbf{B} \quad \text{and} \quad \mathbf{A}_0 = \mathbf{A}(\lambda)|_{\lambda=0} = \mathbf{A}(0).$$

Solving the eigenvalue problem of the companion reference system, the 10 eigenvalues are shown in Table 1.

Conventional iterative eigensolution method can be used to solve the eigenvalues and eigenvectors of the frequency dependent system with tight error control $\varepsilon = 10^{-8}$ and the eigenvalues are computed as shown in Table 2.

Based on the theoretical development of the Perturbative Iteration method discussed in Section 3, a computer code has been written to perform the iterations to compute the eigenvalues and eigenvectors of the frequency dependent system. For the first 4 modes, the iterations based on the proposed PI algorithm produce numerically exact results both for eigenvalues and eigenvectors. The iteration results for the eigenvalues are shown Fig. 3.

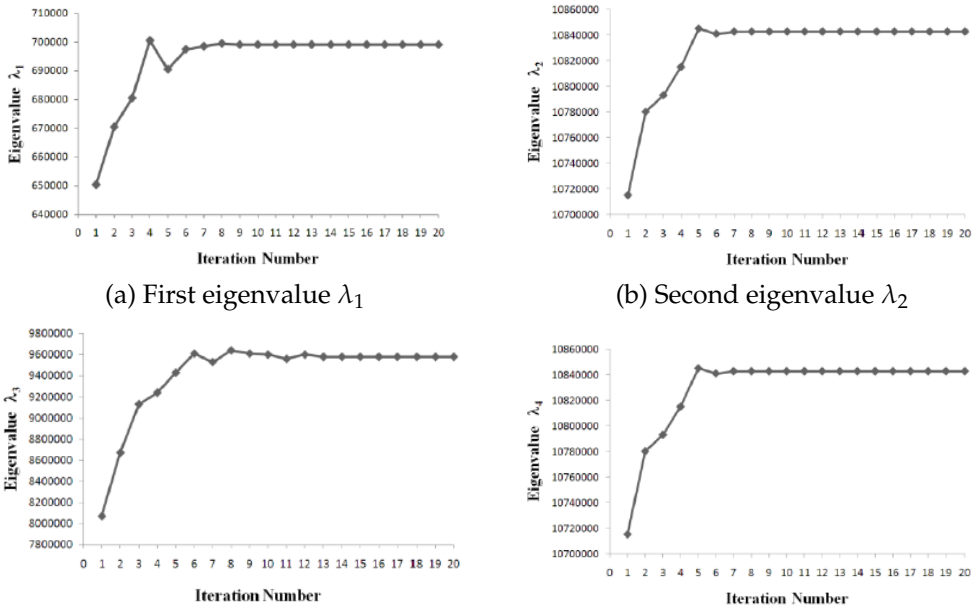
As can be observed, convergence of the iterative process of the proposed PI method is generally very fast and only a few iterations are required before very accurate results are obtained. To further assess the practicality of the proposed new method, a practical

Table 1: Eigenvalues of the companion reference system.

Mode No.	1	2	3	4	5
Eigenvalue	650534.7411	3236116.562	8069734.543	10715001.30	25918539.62
Mode No.	6	7	8	9	10
Eigenvalue	40000000.00	43937413.21	50767873.90	60000000.00	66704786.09

Table 2: Eigenvalues of the frequency dependent system.

Mode No.	1	2	3	4	5
Eigenvalue	699101.6179	3355388.282	9579522.282	10842730.46	30315672.19
Mode No.	6	7	8	9	10
Eigenvalue	40000000.00	44090582.70	60258874.04	79652142.57	88366951.34



(a) First eigenvalue λ_1 (b) Second eigenvalue λ_2

(c) Third eigenvalue λ_3 (d) Fourth eigenvalue λ_4

Figure 3: Iteration results showing convergence of the lower 4 modes.

frequency dependent sandwich beam structure with double layers of viscoelastic core shown in Fig. 4 is used as a more realistic practical case example.

The three constraining layers are assumed to be aluminum with Young's modulus $E = 7.1 \times 10^{10}\text{N/m}^2$ and density $\rho = 2710\text{kg/m}^3$. The viscoelastic cores are ISD-112 material and under room temperature of 24°C, the shear modulus (N/m^2) and the loss factor are assumed to be expressed as

$$G(\omega) = 9.94 \times 10^5 \times e^{0.6937\times\ln(\omega/2\pi)-2.6962}, \tag{4.2a}$$

$$\eta(\omega) = e^{-0.08807\times\ln(\omega/2\pi)+0.60503}, \tag{4.2b}$$

for the convenience of this study though in practice, $G(\omega)$ and $\eta(\omega)$ are obtained from the given nomogram as shown in Fig. 1. In fact, the relationships of (4.2a) and (4.2b) were obtained from the data given in the frequency range up to 1500Hz using curve-fitting analysis.

With these given material properties and structural geometries, finite element modeling was then conducted by using 30 solid elements with 8 nodes for each layer as shown in Fig. 4. Since each layer of surface has 45 nodes and each node is assumed to have 3 degrees of freedom, the total number of degrees of freedom specified in the finite element model for all the 6 layers of surface therefore becomes $3 \times 45 \times 6 = 810$. During the modeling, the shear modulus and loss factor for the viscoelastic cores were assumed to be taken at a reference frequency $f = 100$Hz. The thus established mass $[M]$ and stiffness $[K]$ and structural damping $[D]$ matrices are then used to obtain the eigenvalues and eigenvectors of the reference companion system by solving the com-

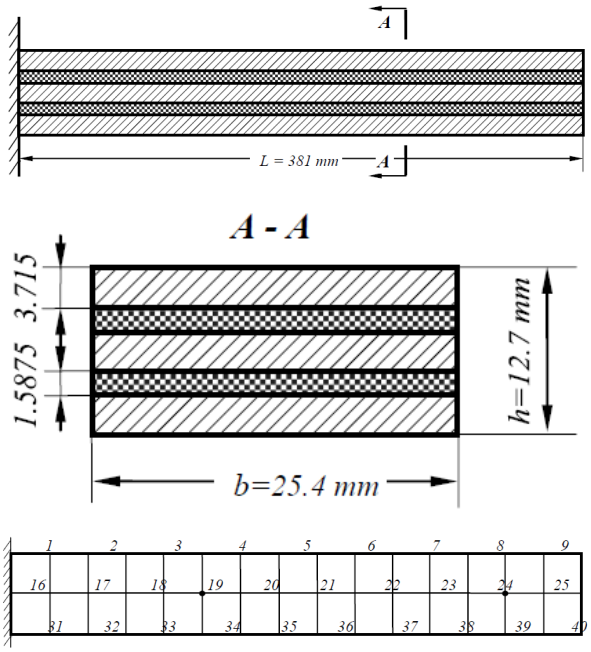


Figure 4: A multilayer sandwiched beam structure with viscoelastic damping.

plex eigenvalue problem of (3.2), where $\mathbf{B}_0 = [M]$ and $\mathbf{A}_0 = [K] + \mathbf{i}[D]$ ($\mathbf{i}$ is the complex notation). The first 5 eigenvalues (in terms of natural frequencies and damping loss factors) of the companion system are shown in Table 3 and Table 4 together with the exact eigenvalues of the frequency dependent system which are obtained using conventional iterative eigensolution on mode by mode basis.

In the case of complete modes where all modes of the companion reference system are included ($N = 810$ in Eq. (3.16a)), the proposed PI method produces exact eigenvalues (and corresponding exact eigenvectors) as shown in Table 3 and Table 4 for the lower 5 modes of interest where accuracy control parameters was given as $\varepsilon = 10^{-8}$. The system considered here is a damped system and the eigenvalues are hence complex which lead to natural frequencies and damping loss factors [21]. For the first mode, iteration results are shown in Fig. 5 and Fig. 6 which demonstrate that convergence for this mode is quite fast. In general, only a few iterations are required before very accurate results are obtained.

In practice, one may not have all the modes of the companion reference system but only a few of the lower M ($M < N$) modes are available, in this case, the proposed PI method can still be formulated by summing up all the available M modes (setting $N = M$ in Eq. (3.16a)). However, the formulation will not be exact in this case due to truncation of higher modes which are not included and thus the computed eigenvalues will be in error as shown in Table 3 and Table 4. In the case of $M = 20$, the errors caused by truncation are quite small as far as practical applications are concerned. Though eigenvectors are not compared due to its large number of elements involved, in the case of complete modes where eigenvalues computed based the proposed PI

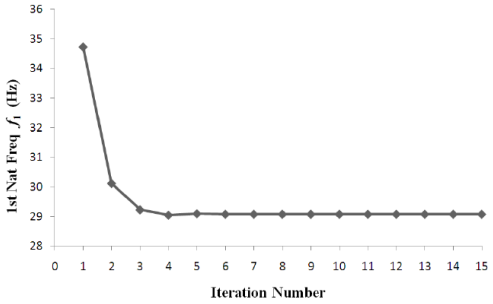


Figure 5: Iteration results for the natural frequency of mode 1.

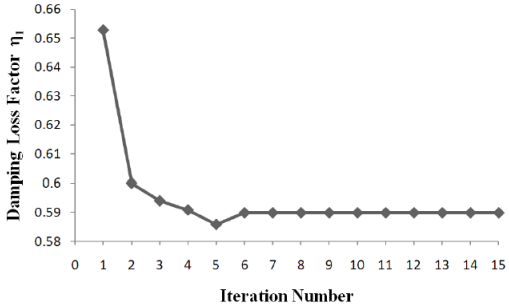


Figure 6: Iteration results for the damping loss factor of mode 1.

method are numerically exact, the corresponding eigenvectors are also numerically exact. On the other hand, when incomplete modes of the companion reference system are used, similar eigenvector error percentage in vector norm to that of eigenvalue has been observed for the various cases discussed.

By employing the eigenproperties of the companion reference system, the proposed PI method progressively computes the required eigenvalues and eigenvectors of interest accurately and efficiently. In the case of complete modes, the computed eigenvalues and eigenvectors are numerically exact while in the case of incomplete modes, the results obtained are very accurate. From computational efficiency point of

Table 3: Natural frequencies of the first 5 modes of the system (Hz).

Mode No.	Companion System	Exact (conventional)	PI (all modes)	PI (20 modes)	Percentage Error
1	34.727373	29.082119	29.082119	29.123871	0.143566%
2	140.67897	152.99209	152.99209	152.73121	-0.17052%
3	342.82570	390.25261	390.25261	389.36512	-0.22741%
4	659.37501	747.93604	747.93604	746.05321	-0.25174%
5	1117.7161	1235.5445	1235.5445	1238.5863	0.246191%

Table 4: Damping loss factors of the first 5 modes of the system.

Mode No.	Companion System	Exact (conventional)	PI (all modes)	PI (20 modes)	Percentage Error
1	0.65271813	0.58985995	0.58985995	0.57900131	-1.84087%
2	0.52014321	0.40644844	0.40644844	0.41250012	1.48892%
3	0.43219821	0.34542296	0.34542296	0.32901787	-4.74927%
4	0.32418882	0.27482320	0.27482320	0.25501621	-7.20717%
5	0.32666112	0.22023118	0.22023118	0.23842914	8.26313%

view, proposed PI method requires far less CPU time than the conventional iterative full eigensolution method. For the 5 modes considered and for the same termination criterion of $\varepsilon = 10^{-8}$, only about 2.5% of CPU time is required when PI method is used as compared with conventional iterative full eigensolution method in the case of complete modes. The computational time required for the complete iterative full eigensolution for the first 5 modes in this case based on Fujitsu Lifebook S Series is 126 minutes. By using the proposed method, the computational time required is 3 minutes and 6 seconds. The convergence of the PI method is generally very fast and only few iterations are required before accurate results are obtained.

5 Limitation and applications of the proposed method

Having developed the mathematical formulation and numerical implementation of the proposed method, it is however worth mentioning that there does exist some limitations on the applicability of the proposed method. It would be customary that once an iterative procedure is proposed, a rigorous proof on the convergence of the procedure is in order, as those presented in [24]. However, the iterative procedure developed in this paper is mathematically based on eigensensitivity analysis and as a result, converegence is not always guaranteed. To ensure convergence, the difference between the actual model and the reference model should be small, that is

$$\frac{\|\mathbf{A}(\lambda)\| - \|\mathbf{A}_0\|}{\|\mathbf{A}_0\|} \ll 1 \quad \text{and} \quad \frac{\|\mathbf{B}(\lambda)\| - \|\mathbf{B}_0\|}{\|\mathbf{B}_0\|} \ll 1. \tag{5.1}$$

Classical eigensensitivity analysis enables us to predict the changes in eigenvalues and eigenvectors of interest based on the eigenvalues and eigenvectors of the current system and the changes to system matrices without solving the eigenvalue problem of the modified system [25]. However, such prediction is never exact but only accurate to a first order provided the changes made satisfy (5.1). The novelty of the present proposed algorithm is that it keeps updating the eigensensitivity using iterations and eventually derives the exact eigenvalue and eigenvector of interest, when the difference between the actual system associated with the frequency of interest and its reference system satisfies (5.1) and hence the iteration process converges. On the other hand, in the practical case where incomplete modes are available, provided sufficient number of modes are included, the proposed algorithm becomes very accurate. Though rigorous error analysis in this case is difficult, the existence of the squared frequency separation $1/(\overline{\lambda}_i - \lambda_r^{(k)})$ term in (3.16a) means that the contributions of higher modes away from the mode of interest become increasingly negligible and hence the truncation error tends to be small. Finally, for repeated modes, eigensensitivities only exist on normalized non-physical space [26] and in this case, the proposed iteration algorithm cannot be used for the particular mode(s). For other modes which are non-repeating, the proposed method can still be applied.

The proposed method is ideally suited for applications where eigenvalues and eigenvectors of modified systems are of interest. These problems arise in structural

dynamics modification prediction and damped eigenvalue problems, among many others. In structural dynamics re-analysis, structural modifications are often made and changes of vibration properties due to these modifications are often sought. In this case, a complete eigensolution *ab* initia is unnecessary and the proposed method can be best employed to solve the problem. In damped vibration problems, through complex eigensoltuion techniques can be used, it is often the case that onw solves the undamped eigenvalue problem first and then, together with the damping matrix, we predict the damping ratios and complex damped eigenvectors. Recognising the fact that the additional damping matrix is usually small as compared with stiffness matrix in Euclidean norm, the proposed method can be ideally applied to computed damped eigenvalues and eigenvectors of interest.

The method has been developed with the practical application cases of structural modification analysis and eigenproperties of damped systems in mind. For these group of applications, the starting reference systems are always available which is the current design in the case structural modification analysis and the undamped system mass and stiffness matrices in the case of damped eigenvalue analysis. However, the method is by no means limited to those applications since mathematically, it is a very general method. For general applications, if one has some priori knowledge about where the natural frequency of interest is, then a good starting companion reference system can be made and the method should work nicely. If, on the other hand, no such information is available, a starting value of frequency estaimate will converge to the mode which is geometrically closest to that starting value.

6 Concluding remarks

Many problems encountered in engineering can be formulated as frequency dependent eigenvalue problems and such eigenvalue problems need to be effectively and efficiently solved for structural vibration analysis, dynamics control and stability, operation deflection and stress analysis etc. However, there has not been any existing effective method to date which addresses this important type of engineering problem even though many methods have been developed for conventional eigenvalue problems. The work presented in this paper seeks to provide a new and effective method which can be used to solve frequency dependent eigenvalue problems. Theoretical development of the proposed method has been presented and computational implementation of the method has been discussed. Extensive numerical case studies have been carried out to fully assess the numerical accuracy and computational efficiency. The method has been extended to the practically more realistic case of incomplete modes and very encouraging results have been achieved.

Mathematically, the method is based on certain type of perturbation analysis and because of this, one would expect that the difference in the sense of matrix norm between the target system and the companion reference system should remain small in order to ensure convergence. Though convergence has been achieved for the cases

which are reported in this paper, in the case of mass-spring system, for some higher modes, convergence was found to be an issue due to the large difference between the target system and the companion reference system. To ensure convergence in this case, there is a need to compute eigenvalues and eigenvectors of an intermediate companion reference system first using the proposed PI algorithm and then, using the intermediate companion reference system as a new companion reference system to compute the eigenvalue and eigenvectors of interest of target system. This will add extra computational cost but as compared with the conventional iterative full eigensolution method, the proposed new method remains computationally efficient.

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