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Group Invariant Solutions of the Full Plastic Torsion of Rod with Arbitrary Shaped Cross Sections

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Abstract. Based on the theory of Lie group analysis, the full plastic torsion of rod with arbitrary shaped cross sections that consists in the equilibrium equation and the non-linear Saint Venant-Mises yield criterion is studied. Full symmetry group admitted by the equilibrium equation and the yield criterion is a finitely generated Lie group with ten parameters. Several subgroups of the full symmetry group are used to generate invariants and group invariant solutions. Moreover, physical explanations of each group invariant solution are discussed by all appropriate transformations. The methodology and solution techniques used belong to the analytical realm.

AMS subject classifications: 74C05, 76M60

Key words: Lie group analysis, group invariant solution, full plastic torsion, yield criterion.

1 Introduction

Lie group analysis is a very important tool in the study of invariant solutions of differential equation. It firstly appeared as an independent research in the Norwegian mathematician Lie's work [1]. The powerfulness of Lie group analysis has been extensively used to find analytic solutions of differential equations. Many scholars have made great efforts in this area [2–7].

In the theory of plasticity, Annin [7, 8] first solved the isothermal flow of ideally rigid-plastic medium with the von Mises yield criterion using Lie-Ovsiannikov group analysis, and he has also found a Lie group of point transformations that admitted by a system of equations of spatial flows. Some results of studying the group properties of equations of isothermal plastic flow of anisotropic and inhomogeneous media were obtained by Senashov [9]. Leonova [10] has studied group invariant solutions of

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equations of visco-plasticity and thermo-visco-plasticity. From the existing literature, we found the application of Lie group analysis is very wide [11–13].

Most classical literatures of solid mechanics [14, 15] describe the torsion of rod, but they are limited to discuss torsion of oval or rectangular rod. In this paper, we study on rod with arbitrary shaped cross sections. We obtain the symmetry group and classify the infinitesimal operators for the full plastic torsion of rod with arbitrary shaped cross sections.

2 Computational method of Lie group analysis

Let us now formulate the computational method of Lie group analysis. Suppose $x = (x_1, x_2, \cdots, x_n)$ is the independent variable, and $u = (u_1, u_2, \cdots, u_m)$ is the dependent variable. We consider the space M with coordinates x, u and the system of differential equations

$$F(F_1, F_2, \cdots, F_r) = 0, \qquad F_i = F_i(x, u, u^{(1)}, \cdots, u^{(s)}). \tag{2.1}$$

Here $u^{(k)}$ is totality of all partial derivatives of order k of functions u_α with respect to x_j, with $i \le r$, $k \le s$, $\alpha \le m$ and $j \le n$.

Infinitesimal operator is

$$X = \xi^i(x, u)\partial_{x_i} + \eta^j(x, u)\partial_{u_j} \left(\begin{array}{c} i = 1, \cdots, n \\ j = 1, \cdots, m \end{array} \right), \tag{2.2}$$

where ξ^i, η^j are tangent vector fields. X defines the group G. Then

$$XF|_M = 0, \tag{2.3}$$

where the notation $|_M$ means evaluated on M.

Symbol $p^\alpha_{i_1,\cdots,i_k}$ corresponds to the derivative

$$p^\alpha_{i_1,\cdots,i_k} = \frac{\partial^k u_\alpha}{\partial x_{i_1} \cdots \partial x_{i_n}}. \tag{2.4}$$

We consider the space $\widetilde{M}$ with coordinates x, u, $u^{(1)}$,$\cdots$, $u^{(k)}$. The k-th prolongation of infinitesimal operator is

$$X_k = X + \zeta^\alpha_i \partial_{p^\alpha_i} + \cdots + \zeta^\alpha_{i_1,\cdots,i_k} \partial_{p^\alpha_{i_1,\cdots,i_k}}, \tag{2.5a}$$

$$\zeta^\alpha_{i_1,\cdots,i_s} = D_{i_s}(\zeta^\alpha_{i_1\cdots i_{s-1}}) - p^\alpha_{\beta i_1 \cdots i_{s-1}} D_{i_s}(\xi^\beta), \tag{2.5b}$$

$$D_{i_s} = \partial_{x_{i_s}} + p^\alpha_{i_s}\partial_{u_\alpha} + p^\alpha_{i_s j_1}\partial_{p^\alpha_{j_1}} + \cdots + p^\alpha_{i_s j_1 \cdots j_{s-1}} \partial_{p^\alpha_{j_1 \cdots j_{s-1}}}. \tag{2.5c}$$

Indices take the following values: $\alpha = 1 \cdots m$, $\beta, i, i_1 \cdots i_k = 1, \cdots, n$; $s \le k$. Repeated indices mean series sum over this index.Consequently, we can get

$$X_k F|_{\widetilde{M}} = 0. \tag{2.6}$$

X_k defines the group $\widetilde{G}$. H is a subgroup of $\widetilde{G}$. $J_1, \cdots, J_t$ are invariants of group H. Suppose

$$\text{rank}\Big[\frac{\partial J_\alpha}{\partial u_\beta}\Big] = m.$$

Then group H has m invariants $J_1, \cdots, J_m$, which are independent of each other. They are functions of $u_1, \cdots, u_m$. The other invariants of group H are the functions of $x_1, \cdots, x_n$. There exists a function

$$J_k = \Phi_k(J_{m+1}, \cdots, J_t), \quad k = 1, \cdots, m. \tag{2.7}$$

Thus, we can obtain the invariant solution.

3 Full plastic torsion of rod with arbitrary shaped cross sections

Let us consider the full plastic torsion problem of rod with arbitrary shaped cross sections obeying the Saint Venant-Mises yield criterion. This problem can be abstracted into the equilibrium equation and the non-linear Saint Venant-Mises yield criterion that defines condition on the second invariant of the stress tensor

$$\begin{cases} \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} = 0, \\ \tau_{xz}^2 + \tau_{yz}^2 = k^2, \end{cases} \tag{3.1}$$

where τ_{xz}, τ_{yz} are components of stress tensor, and k is a constant of plasticity.

We introduce a torsion stress function u,

$$\tau_{xz} = \frac{\partial u}{\partial y}, \qquad -\tau_{yz} = \frac{\partial u}{\partial x}. \tag{3.2}$$

Suppose

$$x_1 = x, \qquad x_2 = y, \qquad p_1 = \frac{\partial u}{\partial x_1}, \qquad p_2 = \frac{\partial u}{\partial x_2}. \tag{3.3}$$

The equilibrium equation transforms to following form

$$u_{x_1}^2 + u_{x_2}^2 = k^2 \quad \text{or} \quad p_1^2 + p_2^2 = k^2. \tag{3.4}$$

The infinitesimal operator of Eq. (3.4) is

$$X = \xi^1(x_1, x_2, u)\frac{\partial}{\partial x_1} + \xi^2(x_1, x_2, u)\frac{\partial}{\partial x_2} + \eta(x_1, x_2, u)\frac{\partial}{\partial u}. \tag{3.5}$$

Its prolongation of infinitesimal operator is

$$\widetilde{X} = X + \zeta_1\frac{\partial}{\partial p_1} + \zeta_2\frac{\partial}{\partial p_2}, \tag{3.6}$$

where

$$\zeta_i = \frac{\partial \eta}{\partial x_i} + p_i \frac{\partial \eta}{\partial u} - p_\beta \Big(\frac{\partial \xi^\beta}{\partial x_i} + p_i \frac{\partial \xi^\beta}{\partial u} \Big). \tag{3.7}$$

Prolongation of infinitesimal operator acts on the manifold of Eq. (3.4). We can get equation

$$p_1 \zeta_1 + p_2 \zeta_2 = 0. \tag{3.8}$$

Take ζ_i into Eq. (3.8), and consider $p_1^2 + p_2^2 = k^2$, then we can get

$$\begin{aligned} &\Big[k^2 \Big(\frac{\partial \eta}{\partial u} - \frac{\partial \xi^2}{\partial x_2} \Big) + p_1 \Big(\frac{\partial \eta}{\partial x_1} - k^2 \frac{\partial \xi^1}{\partial u} \Big) - p_1^2 \Big(\frac{\partial \xi^1}{\partial x_1} - \frac{\partial \xi^2}{\partial x_2} \Big) \Big]^2 \\ =&(k^2 - p_1^2) \Big[k^2 \frac{\partial \xi^2}{\partial u} - \frac{\partial \eta}{\partial x_2} + p_1 \Big(\frac{\partial \xi^2}{\partial x_1} + \frac{\partial \xi^1}{\partial x_2} \Big) \Big]^2. \end{aligned} \tag{3.9}$$

Expand the Eq. (3.9) with the power series of parameter p_1, we can sort the equation-Set the coefficients of each power series of parameter p_1 to zero, we get the decision system of equations

$$\frac{\partial \xi^1}{\partial x_1} - \frac{\partial \xi^2}{\partial x_2} = 0, \qquad \frac{\partial \xi^2}{\partial x_1} + \frac{\partial \xi^1}{\partial x_2} = 0, \tag{3.10a}$$

$$k^2 \frac{\partial \xi^2}{\partial u} - \frac{\partial \eta}{\partial x_2} = 0, \qquad \frac{\partial \eta}{\partial x_1} - k^2 \frac{\partial \xi^1}{\partial u} = 0, \tag{3.10b}$$

$$\frac{\partial \eta}{\partial u} - \frac{\partial \xi^2}{\partial x_2} = 0. \tag{3.10c}$$

Computing the integral of the decision system of equations, we can obtain the coefficients of infinitesimal operator of group that admitted by the equilibrium equations

$$\xi^1 = \frac{1}{k^2}(c_1 u + c_6) x_1 + \frac{1}{2} c_2 \Big(x_1^2 - x_2^2 + \frac{u}{k^2} \Big) + c_3 x_1 x_2 + \frac{c_7}{k^2} u - c_5 x_2 + c_8, \tag{3.11a}$$

$$\xi^2 = \frac{1}{k^2}(c_1 u + c_6) x_1 + \frac{1}{2} c_3 \Big(- x_1^2 + x_2^2 + \frac{u}{k^2} \Big) + c_2 x_1 x_2 + \frac{c_4}{k^2} u + c_5 x_1 + c_9, \tag{3.11b}$$

$$\eta = \frac{1}{2} c_1 \Big(x_1^2 + x_2^2 + \frac{u^2}{k^2} \Big) + (c_2 u + c_7) x_1 + (c_3 u + c_4) x_2 + \frac{c_6}{k^2} u + c_{10}, \tag{3.11c}$$

where $c_i, i = 1, \cdots, 10$ are integral constants

Therefore, the above three Eqs. (3.11a), (3.11b) and (3.11c) can define the 10-

dimensional symmetry group G_{10}. The bases for corresponding Lie algebra are

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial u}, \\
&X_4 = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}, X_5 = \frac{u}{k^2}\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial u}, \\
&X_6 = \frac{u}{k^2}\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial u}, \qquad X_7 = \frac{x_1}{k^2}\frac{\partial}{\partial x_1} + \frac{x_2}{k^2}\frac{\partial}{\partial x_2} + \frac{u}{k^2}\frac{\partial}{\partial u}, \\
&X_8 = \frac{ux_1}{k^2}\frac{\partial}{\partial x_1} + \frac{ux_2}{k^2}\frac{\partial}{\partial x_2} + \frac{1}{2}\Big(x_1^2 + x_2^2 + \frac{u^2}{k^2}\Big)\frac{\partial}{\partial u}, \\
&X_9 = \frac{1}{2}\Big(x_1^2 - x_2^2 + \frac{u^2}{k^2}\Big)\frac{\partial}{\partial x_1} + x_1x_2\frac{\partial}{\partial x_2} + ux_1\frac{\partial}{\partial u}, \\
&X_{10} = x_1x_2\frac{\partial}{\partial x_1} + \frac{1}{2}\Big(-x_1^2 + x_2^2 + \frac{u^2}{k^2}\Big)\frac{\partial}{\partial x_2} + ux_2\frac{\partial}{\partial u}.
\end{aligned}
$$

Then, we can get the invariants of each operator, and the group invariant solutions can be derived by the invariants with Eq. (3.4). See Table 1.

We remark that invariants corresponding to operators X_3, X_9, X_{10} do not meet the existence conditions of group invariant solutions, because

$$l = \text{rank}\Big[\frac{\partial J_\alpha}{\partial u}\Big] \neq m.$$

$u_0|_{X_i}$ are integral constants which determined by the boundary conditions. As an example, for the solid section rod, no surface force on side, boundary condition is

$$u|_S = 0.$$

on end-face boundary condition is

$$2\iint udxdy = M_l,$$

where M_l is plastic limit torque.

According to the infinitesimal operators, we can clearly see their transformations, and the transformations are also the cause of displacements. See Table 2.

4 Conclusions and discussion

We have completely solved the full plastic torsion problem of rod with arbitrary shaped cross sections using Lie group analysis. The 10-dimensional Lie algebra that is admitted by the equilibrium equation and the yield criterion, and group invariant solutions that relative to several sub-algebras are given. We build a theoretical framework of Lie group analysis. In this framework, we can get many such types of motion for one time only.

Table 1: Invariants and group invariant solutions of symmetry group.

Infinitesimal Operator	Invariant		Group Invariant Solution
	J_1	J_2	
X_1	u	x_2	$\pm kx_2 + u_0\vert_{X_1}$
X_2	u	x_1	$\pm kx_1 + u_0\vert_{X_2}$
X_3	x_1	x_2	Null
X_4	u	$x_1^2 + x_2^2$	$\pm k\sqrt{x_1^2 + x_2^2} + u_0\vert_{X_4}$
X_5	$-k^2x_1^2 + u^2$	x_2	$\pm\sqrt{k^2x_1^2 + (kx_2 + u_0\vert_{X_5})^2}$
X_6	$-k^2x_2^2 + u^2$	x_1	$\pm\sqrt{k^2x_2^2 + (kx_1 + u_0\vert_{X_6})^2}$
X_7	$\frac{u}{x_1}$	$\frac{x_2}{x_1}$	$\pm kx_2 + u_0\vert_{X_7}$
X_8	$\frac{-k^2x_1^2-k^2x_2^2+u^2}{x_1}$	$\frac{x_1}{x_2}$	$\pm k\sqrt{x_1^2 + x_2^2} + u_0\vert_{X_8}$
X_9	$\frac{k^2x_1^2+k^2x_2^2-u^2}{k^2x_2}$	$\frac{u}{x_2}$	Null
X_{10}	$\frac{k^2x_1^2+k^2x_2^2-u^2}{k^2x_1}$	$\frac{u}{x_1}$	Null

Table 2: Types of transformations and physical explanation of symmetry group.

Infinitesimal operator	Type of transformations	Physical explanation
X_1, X_2, X_3	$x_1' = x_1 + a_1$, $x_2' = x_2 + a_2$, $u = u + a_3$	Translation transformations
X_4	$x_1' = x_1 \cos a_4 + x_2 \sin a_4$, $x_2' = -x_1 \sin a_4 + x_2 \cos a_4$	Rotation transformation
X_5, X_6	$u' = \frac{u}{k^2}\mathrm{ch}a_5 + x_1\mathrm{sh}a_5$, $x_1' = \frac{u}{k^2}\mathrm{sh}a_5 + x_1\mathrm{ch}a_5$ $u' = \frac{u}{k^2}\mathrm{ch}a_6 + x_2\mathrm{sh}a_6$, $x_2' = \frac{u}{k^2}\mathrm{sh}a_6 + x_2\mathrm{ch}a_6$	Hyperbolic rotations transformation
X_7	$x_1' = \frac{x_1}{k^2}\exp a_7$, $x_2' = \frac{x_2}{k^2}\exp a_7$, $u' = \frac{u}{k^2}\exp a_7$	Stretching transformation
X_8, X_9, X_{10}	$\frac{\partial x^{i'}}{x^i}\Gamma^i_{kl} = \Gamma^{i'}_{n'p'}\frac{\partial x^{n'}}{\partial x^k}\frac{\partial x^{p'}}{\partial x^l} + \frac{\partial^2 x^{i'}}{\partial x^k \partial x^l}$ conformal transformation of coordinate	Conformal transformation

According to the infinitesimal operators, we define and classify the types of motion. Furthermore, solving types of motion are exhaustive, because we do not introduce any additional assumption. In addition, the torsion stress function is smooth. For example, $u|_{X_1}$, $u|_{X_2}$ represent the cause of translational motion, $u|_{X_4}$ represents the cause of rotation motion, $u|_{X_5}$, $u|_{X_6}$ represent the cause of hyperbolic rotations transformation, $u|_{X_7}$ represents the cause of stretching transformation, $u|_{X_8}$ represents the cause of conformal transformation to our best knowledge of the state-of-the-art. For the full plastic torsion of rod with arbitrary shaped cross sections, the use of conformal transformation to solve the problem was not found in literature. Although Sokolnikoff [14,16] used conformal transformation to solve the torsion of rod whose cross sections are bounded by two intersecting symmetric circular arcs, but it is limited to simply connected region.

Group invariant solutions have great values. First of all, each group invariant solution represents a basic type of motion, and totally meets the boundary conditions or initial conditions. The analyticity of group invariant solution also enables us to directly determine its other features. Secondly, the group invariant solutions may be used to benchmark to test numerical algorithm.

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