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# The Multi-Step Differential Transform Method and Its Application to Determine the Solutions of Non-Linear Oscillators

Vedat Suat Ertürk[1,*], Zaid M. Odibat[2] and Shaher Momani[3]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Ondokuz Mayis University, 55139, Samsun, Turkey*
[2] *Prince Abdullah Bin Ghazi Faculty of Science and IT, Al-Balqa Applied University, Al-Salt 19117, Jordan*
[3] *The University of Jordan, Faculty of Science, Department of Mathermatics, Amman 1194, Jordan*

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**Abstract.** In this paper, a reliable algorithm based on an adaptation of the standard differential transform method is presented, which is the multi-step differential transform method (MSDTM). The solutions of non-linear oscillators were obtained by MSDTM. Figurative comparisons between the MSDTM and the classical fourth-order Runge-Kutta method (RK4) reveal that the proposed technique is a promising tool to solve non-linear oscillators.

**AMS subject classifications**: 74H15, 35E15, 37M05

**Key words**: Non-linear oscillatory systems, differential transform method, numerical solution.

## 1 Introduction

Vibration problems and most of scientific problems in mechanics are naturally nonlinear. The equations modeling all these phenomena and problems are either ordinary or partial differential equations.Most of them do not have any analytical solution excepting a restricted set of these problems. Some are solved via the analytical perturbation method [1], whereas some of them are solved by numerical techniques. In many case studies, similarity transformations are used to reduce the governing differential equations into an ordinary nonlinear differential equation. In most cases, these equations do not have analytical solution. Therefore, these equations should be solved via spec-

*Corresponding author.
*URL:* `http://www2.omu.edu.tr/akademikper.asp?id=156`
*Email:* vserturk@yahoo.com (V. S. Ertürk), z.odibat@gmail.com (Z. M. Odibat), shahermm@yahoo.com (S. Momani)

cial techniques. In last years, new methods were used by some researchers to solve these sorts of problem [2–5].

Integral transform methods like the Laplace and the Fourier transform methods are extensively used in engineering problems. By using these methods, differential equations are transformed into algebraic equations which are easier to cope with. In fact, integral transform methods are more complex and difficult when applying to nonlinear problems. A different dealing method to solve non-linear initial value problems is the MSDTM [6]. Our motivation is to concentrate on the applications of the multistep differential transform method (MSDTM).It should be mentioned that one of the main advantages of the MSDTM is its ability in providing us a continuous representation of the approximate solution, which allows better information of the solution over the time interval.

On the other side, the Runge-Kutta method (RK4) will provide solutions in discretized form, only at two ends of the time interval, thereby making it complicated in achieving a continuous representation. We purpose to contrast the effectiveness of MSDTM against the well-known fourth-order Runge-Kutta method.

Nonlinear oscillator models have been extensively encountered in many areas of physics and engineering. These models have meaningful importance in mechanical and structural dynamics for the wide understanding and accurate prediction of motion. Since many practical engineering components comprise of vibrating systems are modeled by using oscillator systems, these systems are important in physics and engineering [7,8].

## 2 Differential transform method

The differential transform technique is one of the semi-numerical analytical methods for ordinary and partial differential equations that uses the form of polynomials as approximations of the exact solutions that are sufficiently differentiable. The basic definition and the fundamental theorems of the differential transform method (DTM) and its applicability for various kinds of differential equations are given in [9–13]. For convenience of the reader, we present a review of the DTM. The differential transform of the $k$th derivative of function $f(t)$ is defined as follows:

$$F(k)=\frac{1}{k!}\Big[\frac{d^k f(t)}{dt^k}\Big]_{t=t_0}, \tag{2.1}$$

where $f(t)$ is the original function and $F(k)$ is the transformed function. The differential inverse transform of $F(k)$ is defined as

$$F(t)=\sum_{k=0}^{\infty}F(k)(t-t_0)^k. \tag{2.2}$$

From Eqs. (2.1) and (2.2), we get

$$f(t)=\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\frac{d^k f(t)}{dt^k}\Big|_{t=t_0}, \tag{2.3}$$

Table 1: Operations of differential transformation.

| Original function | Transformed function |
|---|---|
| $f(t)=u(t)\pm v(t)$ | $F(k)=U(k)\pm V(k)$ |
| $f(t)=\alpha u(t)$ | $F(k)=\alpha U(k)$ |
| $f(t)=u(t)v(t)$ | $F(k)=\sum_{l=0}^{k} U(l)\pm V(k-l)$ |
| $f(t)=\dfrac{du(t)}{dt}$ | $F(k)=(k+1)U(k+1)$ |
| $f(t)=\dfrac{d^m u(t)}{dt^m}$ | $F(k)=(k+1)(k+2)\cdots(k+m)U(k+m)$ |
| $f(t)=\int_{t_0}^{t} u(t)dt$ | $F(k)=\dfrac{U(k-1)}{k},\quad k\geq 1$ |
| $f(t)=t^m$ | $F(k)=\delta(k-m)=\begin{cases}1, & k=m,\\ 0, & k\neq m,\end{cases}$ |
| $f(t)=\exp(\lambda,t)$ | $F(k)=\dfrac{\lambda^k}{k!}$ |
| $f(t)=\sin(\omega t+\alpha)$ | $F(k)=\dfrac{w^k}{k!}\sin\left(\dfrac{\pi k}{2}+\alpha\right)$ |
| $f(t)=\cos(\omega t+\alpha)$ | $F(k)=\dfrac{w^k}{k!}\cos\left(\dfrac{\pi k}{2}+\alpha\right)$ |
| $h(t)=\dfrac{f(t)}{g(t)}$ | $K(k)=\dfrac{1}{G(0)}\Big[F(k)-\sum_{m=0}^{k-1} H(m)G(k-m)\Big]$ |
| $f(t)=[g(t)]^b$ | $F(k)=\begin{cases}G(0), & k=0,\\ \sum_{m=1}^{k}\dfrac{(b+1)m-k}{kG(0)}G(m)F(k-m), & k\geq 1.\end{cases}$ |

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original function. For implementation purposes, the function $f(t)$ is expressed by a finite series and Eq. (2.2) can be written as

$$f(t)\approx\sum_{k=0}^{N}(t-t_0)^k. \tag{2.4}$$

Here $N$ is decided by the convergence of natural frequency. The fundamental operations performed by differential transform can readily be obtained and are listed in Table 1. The main steps of the DTM, as a tool for solving different classes of nonlinear problems, are the following. First, we apply the differential transform (2.1) to the given problem (integral equation, ordinary differential equation or partial differential equations), then the result is a recurrence relation. Second, solving this relation and using the differential inverse transform (2.2), we can obtain the solution of the problem.

## 3 Multi-step differential transform method

Although the DTM is used to provide approximate solutions for a wide class of nonlinear problems in terms of convergent series with easily computable components, it has some drawbacks: the series solution always converges in a very small region and it has slow convergent rate or completely divergent in the wider region [14–17]. In this section, we present the multi-step DTM that hase been developed in [6] for the numerical solution of differential equations. For this purpose, we consider the following nonlinear initial value problem

$$f(t,y,y',\cdots,y^{(p)})=0, \tag{3.1}$$

subject to the initial conditions $y^{(0)}=c_k$, for $k=0,1,\cdots,p-1$.

Let $[0,T]$ be the interval over which we want to find the solution of the initial value problem (3.1). In actual applications of the DTM, the approximate solution of the initial value problem (3.1) can be expressed by the finite series

$$y(t)=\sum_{n=0}^{N}\alpha_n t^n, \qquad t\in[0,T]. \tag{3.2}$$

The multi-step approach introduces a new idea for constructing the approximate solution. Assume that the interval $[0,T]$ is divided into $M$ subintervals $[t_{m-1},t_m]$, $m=1,2,\cdots,M$ of equal step size $h=T/M$ by using the nodes $t_m=mh$. The main idea of the multi-step DTM are as follows. First, we apply the DTM to Eq. (3.1) over the interval $[0,t_1]$, we will obtain the following approximate solution

$$y_1(t)=\sum_{n=0}^{K}\alpha_{1n} t^n, \qquad t\in[0,t_1], \tag{3.3}$$

using the initial conditions $y_1^{(k)}=c_k$. For $m\geq 2$ and at each subinterval $[t_{m-1},t_m]$ we will use the initial conditions $y_m^{(k)}(t_{m-1})=y_{m-1}^{(k)}(t_{m-1})$ and apply the DTM to Eq. (3.1) over the interval $[t_{m-1},t_m]$, where $t_0$ in Eq. (2.1) is replaced by $t_{m-1}$ The process is repeated and generates a sequence of approximate solutions $y_m(t)$, $m=1,2,\cdots,M$, for the solution of $y(t)$,

$$y_m(t)=\sum_{n=0}^{K}\alpha_{mn}(t-t_{m-1})^n, \qquad t\in[t_m,t_{m+1}], \tag{3.4}$$

where $N=K\cdot M$. In fact, the multi-step DTM assumes the following solution

$$y(t)=\begin{cases} y_1(t), & t\in[0,t_1],\\ y_2(t), & t\in[t_1,t_2],\\ \vdots & \\ y_M(t), & t\in[t_{M-1},t_M]. \end{cases} \tag{3.5}$$

The new algorithm, multi-step DTM, is simple for computational performance for all values of $h$. It is easily observed that if the step size $h$ is $T$, then the multi-step DTM reduces to the classical DTM. As we will see in the next section, the main advantage of the new algorithm is that the obtained series solution converges for wide time regions.

# 4 Numerical experiments

To demonstrate the applicability of the proposed algorithm as an approximate tool for solving nonlinear oscillatory systems, we apply the proposed algorithm, the multi-step DTM, to four non-linear oscillator equations.

**Example 4.1.** Consider the following non-linear equation

$$y'' + y = -\varepsilon y^2 y', \tag{4.1}$$

subject to the following initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{4.2}$$

This equation can be appropriately called the "unplugged" Van der Pol equation and all of its solutions are expected to oscillate with decreasing amplitude to zero. Momani et al. [18] derived a numerical solution for the above equation using the modified homotopy perturbation method when $\varepsilon = 0.1$. For comparison with the solution obtained in [18] we set the parameter $\varepsilon = 0.1$ in this example.

With $y' = x$, Eq. (4.1) is transformed into the following system of the first-order differential equations

$$\frac{dy}{dt} = x, \qquad \frac{dx}{dt} = -y - \varepsilon y^2 x, \tag{4.3}$$

and the initial conditions $y(0) = 1$ and $y'(0) = 0$ become

$$y(0) = 1, \qquad x(0) = 0. \tag{4.4}$$

In view of the differential transform, given in Eq. (2.1), and the operations of differential transform given in Table 1, applying the differential transform to the system (4.3), we obtain

$$\begin{cases} Y(k+1) = \dfrac{1}{k+1} X(k), \\ X(k+1) = \dfrac{1}{k+1}\Big( -Y(k) - \varepsilon \displaystyle\sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} Y(k_1) Y(k_2 - k_1) X(k - k_2)\Big), \end{cases} \tag{4.5}$$

where $X(k)$ and $Y(k)$ are the differential transforms of $x(t)$ and $y(t)$, respectively. The differential transform of the initial conditions are given by $Y(0) = 1$ and $X(0) = 0$. In view of the inverse differential transform, given in Eq. (2.2), the DTM series solution for the system (4.3) can be obtained as

$$\begin{cases} y(t) = \displaystyle\sum_{n=0}^{N} Y(n) t^n, \\ x(t) = \displaystyle\sum_{n=0}^{N} X(n) t^n. \end{cases} \tag{4.6}$$

Now, according to the multi-step DTM, taking $N = K \cdot M$, the series solution for the system (4.3) is given by

$$y(t) = \begin{cases} \sum_{n=0}^{K} Y_1(n)t^n, & t \in [0, t_1], \\ \sum_{n=0}^{K} Y_2(n)(t-t_1)^n, & t \in [t_1, t_2], \\ \vdots & \\ \sum_{n=0}^{K} Y_M(n)(t-t_{M-1})^n, & t \in [t_{M-1}, t_M], \end{cases} \tag{4.7a}$$

$$x(t) = \begin{cases} \sum_{n=0}^{K} X_1(n)t^n, & t \in [0, t_1], \\ \sum_{n=0}^{K} X_2(n)(t-t_1)^n, & t \in [t_1, t_2], \\ \vdots & \\ \sum_{n=0}^{K} X_M(n)(t-t_{M-1})^n, & t \in [t_{M-1}, t_M], \end{cases} \tag{4.7b}$$

where $Y_i(n)$ and $X_i(n)$ for $i = 1, 2, \cdots, M$ satisfy the following recurrence relations

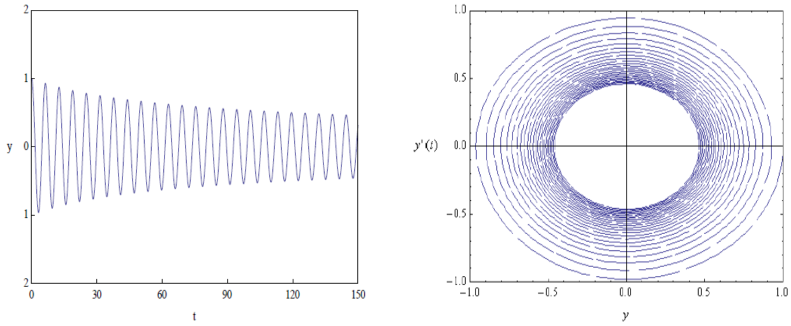
$$\begin{cases} Y_i(k+1) = \dfrac{1}{k+1} X_i(k), \\ X_i(k+1) = -Y_i(k) - \varepsilon \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} Y_i(k_1) Y_i(k_2-k_1) X_i(k-k_2), \end{cases} \tag{4.8}$$

such that $Y_i(0) = Y_{i-1}(0)$ and $X_i(0) = X_{i-1}(0)$. Finally, if we start with $Y(0) = 1$ and $X(0) = 0$, using the recurrence relations given in (4.8), then we can obtain the multi-step solution given in Eqs. (4.7a) and (4.7b). Fig. 1 shows the displacement and phase diagram of the multi step DTM solution to the nonlinear equation (4.1).

Fig. 1 shows that the results of our computations are in excellent agreement with the results obtained by the numerical solution of Momani et al. [18] using homotopy perturbation method method. It is to be noted that the multi-step DTM results are obtained when $K = 10$, $M = 250$ and $T = 150$, and the multi-step DTM results are obtained when $N = 2500$.

In Fig. 2, we give a comparison among the multi-step DTM solution, the DTM solution and RK4 solution for the problem (4.1)-(4.2). Fig. 2 shows the advantage of the multi-step DTM over DTM . One can see that DTM exhibits great error because its solution increasingly goes to infinity.

We determine the accuracy of RK4 for the solution of (4.3) for different time steps. From the results presented in Table 2 we see that the maximum difference between the RK4 solutions on the time steps $h = 0.01$ and $h = 0.001$ is of the order of $10^{-9}$ whereas the maximum difference between the time steps of $h = 0.001$ and $h = 0.0001$ is of the




(a) (b)

Figure 1: (a) Displacement $y$ versus time $t$ and (b) phase plane diagram.

order of $10^{13}$. Although RK4 results are relatively accurate for $h = 0.01$, we choose $h = 0.001$ for its higher precision.

Table 2 shows two sets of MSDTM solutions being compared to RK4 $(h = 0.001)$. The absolute values were obtained to determine its performance against RK4. The first set of result is between a 5 term MSDTM $(h = 0.01)$ and RK4. We could see clearly that the highest error is of . Next, we proceed to compare 5 term MSDTM $(h = 0.001)$ with RK4. The accuracy is strengthened by a maximum error of $10^{-13}$. The results show that MSDTM is an excellent tool in solving the equation considered.

**Example 4.2.** Consider the following initial-value problem

$$y'' + y + 6y^2 + 8y^3 = 0, \tag{4.9}$$

subject to the following initial conditions

$$y(0) = 2, \qquad y'(0) = 0. \tag{4.10}$$

Let $y' = x$. Then Eq. (4.9) can be written as

$$\frac{dy}{dt} = x, \qquad \frac{dx}{dt} = -y - 6y^2 - 8y^3, \tag{4.11}$$

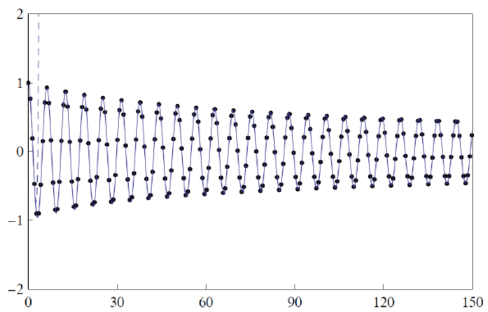



Figure 2: Plots of displacement of $y$ versus time $t$: Dashed line (the DTM solution); Solid line (the multi-step DTM solution, Dotted line (RK4 solution).

Table 2: Absolute differences between RK4 solutions.

| | $\Delta=\lvert RK4_{0.01}-RK4_{0.0001}\rvert$ | | $\Delta=\lvert RK4_{0.001}-RK4_{0.0001}\rvert$ | |
|---|---|---|---|---|
| $t$ | $\Delta x$ | $\Delta y$ | $\Delta x$ | $\Delta y$ |
| 15 | 8.63388E-10 | 6.7354E-10 | 1.03473E-13 | 5.06262E-14 |
| 30 | 3.15983E-10 | 1.86788E-9 | 6.79456E-14 | 1.86684E-13 |
| 45 | 1.31539E-9 | 2.19618E-9 | 9.88098E-14 | 2.46636E-13 |
| 60 | 2.98053E-9 | 9.83527E-10 | 3.02092E-13 | 1.18461E-13 |
| 75 | 3.43937E-9 | 1.436E-9 | 3.60684E-13 | 1.47105E-13 |
| 90 | 1.8496E-9 | 3.74658E-9 | 2.0195E-13 | 3.79141E-13 |
| 105 | 1.15989E-9 | 4.4542E-9 | 1.05915E-13 | 4.62852E-13 |
| 120 | 4.06571E-9 | 2.85524E-9 | 4.06564E-13 | 3.10585E-13 |
| 135 | 5.36705E-9 | 5.40324E-10 | 5.52822E-13 | 2.81442E-14 |
| 150 | 3.98036E-9 | 4.17555E-9 | 4.19054E-13 | 4.11671E-13 |

Table 3: Absolute errors between MSDTM and RK4 solutions.

| | $\Delta=\lvert MSDTM(K=5)_{0.01}-RK4_{0.001}\rvert$ | | $\Delta=\lvert MSDTM(K=5)_{0.001}-RK4_{0.001}\rvert$ | |
|---|---|---|---|---|
| $t$ | $\Delta x$ | $\Delta y$ | $\Delta x$ | $\Delta y$ |
| 15 | 1.0163E-12 | 1.04294E-12 | 4.07452E-14 | 1.31006E-14 |
| 30 | 2.14651E-12 | 5.08191E-13 | 2.24265E-14 | 6.21864E-14 |
| 45 | 2.74858E-12 | 1.321E-12 | 2.24265E-14 | 6.62248E-14 |
| 60 | 1.54521E-12 | 3.38196E-12 | 6.9611E-14 | 4.81837E-14 |
| 75 | 1.27873E-12 | 3.92308E-12 | 9.33975E-14 | 8.10463E-15 |
| 90 | 3.83604E-12 | 2.30871E-12 | 6.86118E-14 | 7.19702E-14 |
| 105 | 4.74898E-12 | 7.51593E-13 | 2.66454E-15 | 1.03362E-13 |
| 120 | 3.43947E-12 | 3.93718E-12 | 8.17124E-14 | 7.9714E-14 |
| 135 | 5.09592E-14 | 5.52297E-12 | 1.20952E-13 | 4.05231E-15 |
| 150 | 3.78625E-12 | 4.36934E-12 | 1.04527E-13 | 8.199E-14 |

which is a system of two equations of order one in two unknown functions. According to the multi-step DTM, the series solution for the system (4.11) is given by

$$(y(t),x(t))=\begin{cases}\sum_{n=0}^{K}(Y_1(n),X_1(n))t^n, & t\in[0,t_1],\\ \sum_{n=0}^{K}(Y_2(n),X_2(n))(t-t_1)^n, & t\in[t_1,t_2],\\ \vdots & \\ \sum_{n=0}^{K}(Y_M(n),X_M(n))(t-t_{M-1})^n, & t\in[t_{M-1},t_M],\end{cases} \tag{4.12}$$

where $Y_i(n)$ and $X_i(n)$, for $i=1,2,\cdots,M$ satisfy the following recurrence relations

$$\begin{cases}Y_i(k+1)=\dfrac{1}{k+1}X_i(k),\\ X_i(k+1)=\dfrac{1}{k+1}\Big(-Y_i(k)-6\sum_{l=0}^{k}Y_i(l)Y_i(k-l)-8\sum_{k_1=0}^{k_2}Y_i(k_1)Y_i(k_2-k_1)Y_i(k-k_2)\Big),\end{cases} \tag{4.13}$$

such that $Y_1(0)=y(0)$, $X_1(0)=x(0)$ and $Y_i(0)=Y_{i-1}(0)$, $X_i(0)=X_{i-1}(0)$, for $i=2,3,\cdots,M$. For this example, we use the values of the stepsize $h$, $K$ and $N$ as 0.04,

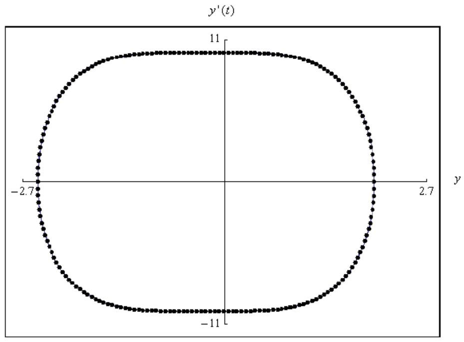



Figure 3: Phase plane for the problem (4.9)-(4.10): Solid line (present solution), Dotted line (RK4 solution).

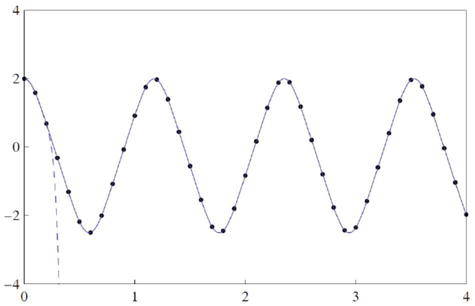



Figure 4: Plots of displacement of $y$ versus time $t$: Dashed line (the DTM solution); Solid line (the multi-step DTM solution, Dotted line (RK4 solution).

10 and 1000, respectively. Fig. 3 shows the comparison between the multi-step DTM solution and the numerical integration results obtained by RK4 method for the phase diagram of the problem (4.9)-(4.10).

Fig. 3. Phase plane for the problem (4.9)-(4.10): Solid line (present solution), Dotted line (RK4 solution). It can be seen from Fig. 3 that the solution obtained by the present method is nearly identical with that given by RK4 method. Also, the results of our computations are in excellent agreement with the results obtained in [19].

The graphs of the solutions corresponding to the multi-step DTM solution, DTM solution and RK4 solution are sketched in Fig. 4. Fig. 4 shows that the multi-step DTM greatly improves the solution obtained by DTM.One can see that DTM exhibits great error because its solution decreasingly goes to to infinity.

Table 4 shows the maximum differences between the RK4 solution on time steps $h = 0.01$ and $h = 0.001$.

Table 5 exhibits two sets of MSDTM solutions being compared to RK4 $(h = 0.001)$. The first set of result is between a 5 term MSDTM $(h = 0.01)$ and RK4. We could see clearly that the highest error is of $10^{-5}$. Next, we proceed to compare a 5 term MSDTM $(h = 0.001)$ with RK4. The accuracy is strengthened by a maximum error of $10^{-8}$.

Table 4: Absolute differences between RK4 solutions.

| | $\Delta = \|RK4_{0.01} - RK4_{0.0001}\|$ | | $\Delta = \|RK4_{0.001} - RK4_{0.0001}\|$ | |
|---|---|---|---|---|
| $t$ | $\Delta x$ | $\Delta y$ | $\Delta x$ | $\Delta y$ |
| 0.5 | 1.13408E-5 | 1.33742E-6 | 1.12236E-9 | 1.21716E-10 |
| 1 | 2.66115E-6 | 4.43673E-6 | 3.66017E-10 | 3.93943E-10 |
| 1.5 | 4.57189E-6 | 7.63335E-6 | 1.95914E-10 | 6.61934E-10 |
| 2 | 7.14694E -6 | 1.08138E-5 | 3.25782E-10 | 9.17359E-10 |
| 2.5 | 3.8657E -5 | 1.29197E-5 | 2.80493E-9 | 1.07407E-9 |
| 3 | 1.31574E -4 | 8.40425E-6 | 1.00006E-8 | 6.98749E-10 |
| 3.5 | 1.82084E-4 | 6.01948E-6 | 1.12236E-9 | 4.00081E-10 |
| 4 | 9.32339E-5 | 2.1128E-5 | 7.07433E-9 | 1.47319E-9 |

Table 5: Absolute differences between MSDTM and RK4 solutions.

| | $\Delta = \|MSDTM(K=5)_{0.01} - RK4_{0.001}\|$ | | $\Delta = \|MSDTM(K=5)_{0.001} - RK4_{0.001}\|$ | |
|---|---|---|---|---|
| $t$ | $\Delta x$ | $\Delta y$ | $\Delta x$ | $\Delta y$ |
| 0.5 | 8.40675E-7 | 9.78464E-8 | 1.13026E-9 | 1.22554E-10 |
| 1 | 1.23449E-7 | 3.40013E-7 | 3.65342E-10 | 3.97121E-10 |
| 1.5 | 1.08766E-6 | 5.66513E-7 | 2.06102E-10 | 6.67335E-10 |
| 2 | 1.62996E-6 | 8.20343E-7 | 3.41373E-10 | 9.25256E-10 |
| 2.5 | 4.88093E-6 | 1.0002E-6 | 2.85221E-9 | 1.08377E-9 |
| 3 | 1.33778E-5 | 5.98731E-7 | 1.01313E-8 | 7.04532E-10 |
| 3.5 | 1.86411E-5 | 8.06935E-7 | 1.38756E-8 | 4.08015E-10 |
| 4 | 9.88639E-6 | 2.65161E-6 | 7.17111E-9 | 1.49914E-9 |

**Example 4.3.** Consider the following nonlinear oscillator [20]

$$y'' + \frac{y}{\sqrt{1+y^2}} = 0, \tag{4.14}$$

with the initial conditions

$$y(0) = B, \qquad y'(0) = 0. \tag{4.15}$$

Let $y' = x$. Then, Eq. (4.14) is transformed into the system of the first-order differential equations

$$\frac{dy}{dt} = x, \qquad \frac{dx}{dt} = -\frac{y}{\sqrt{1+y^2}}, \tag{4.16}$$

and the initial conditions $y(0) = 0$ and $y'(0) = 0$ become

$$y(0) = B, \qquad x(0) = 0. \tag{4.17}$$

In view of the differential transform, given in Eq. (2.1), and the operations of differential transform given in Table 1, applying the differential transform to the system (4.16), we obtain

$$\begin{cases} Y(k+1) = \dfrac{1}{k+1}X(k), \\ X(k+1) = -\dfrac{1}{k+1}H(k), \end{cases} \tag{4.18}$$

where $X(k)$, $Y(k)$ and $H(k)$ are the differential transforms of $x(t)$, $y(t)$ and $h(t) = y(t)/\sqrt{1+y^2(t)}$, respectively.

The differential transform of the initial conditions are given by $Y(0) = B$ and $X(0) = 0$. In view of the inverse differential transform, given in Eq. (2.2), the DTM series solution for the system (4.16) can be obtained as

$$\begin{cases} y(t) = \sum_{n=0}^{N} Y(n)t^n, \\ x(t) = \sum_{n=0}^{N} X(n)t^n. \end{cases} \tag{4.19}$$

Now, according to the multi-step DTM, taking $N = K \cdot M$, the series solution for the system (4.16) is given by

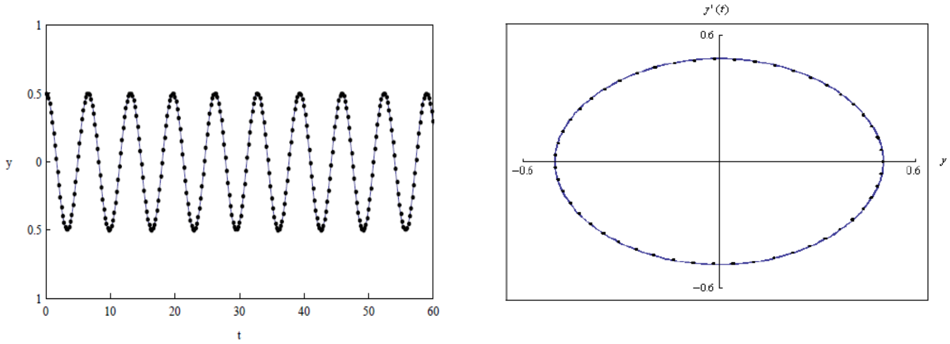
$$y(t) = \begin{cases} \sum_{n=0}^{K} Y_1(n)t^n, & t \in [0, t_1], \\ \sum_{n=0}^{K} Y_2(n)(t-t_1)^n, & t \in [t_1, t_2], \\ \vdots \\ \sum_{n=0}^{K} Y_M(n)(t-t_{M-1})^n, & t \in [t_{M-1}, t_M], \end{cases} \tag{4.20a}$$

$$x(t) = \begin{cases} \sum_{n=0}^{K} X_1(n)t^n, & t \in [0, t_1], \\ \sum_{n=0}^{K} X_2(n)(t-t_1)^n, & t \in [t_1, t_2], \\ \vdots \\ \sum_{n=0}^{K} X_M(n)(t-t_{M-1})^n, & t \in [t_{M-1}, t_M], \end{cases} \tag{4.20b}$$

where $Y_i(n)$ and $X_i(n)$, for $i = 1, 2, \cdots, M$ satisfy the following recurrence relations

$$\begin{cases} Y_i(k+1) = \frac{1}{k+1} X_i(k), \\ X_i(k+1) = -\frac{1}{k+1} H_i(k), \end{cases} \tag{4.21}$$

such that $Y_i(0) = Y_{i-1}(0)$ and $X_i(0) = X_{i-1}(0)$. Finally, if we start with $Y(0) = B$ and $X(0) = 0$, using the recurrence relations given in (4.21), then we can obtain the multi-step solution given in Eqs. (4.20a) and (4.20b). We set the parameter $B = 0.5$ in this example. The values of $K$ and $N$ are taken as 10 and 3000, respectively. Fig. 5 exhibits the comparison between the multi-step DTM solution and the numerical integration results obtained by RK4 method for the displacement and phase diagram of nonlinear equation (4.14) subject to the initial conditions (4.15). It can be seen from Fig. 5 that




(a) (b)

Figure 5: The displacement and phase plane for Example 4.3: Solid line (present solution), Dotted line (RK4 solution).

the solution obtained by the present method is nearly identical with that given by RK4 method.

In Fig. 6, we give a comparison against the multi-step DTM solution, the DTM solution and RK4 solution for the problem (4.14)-(4.15). Fig. 6 shows the advantage of multi-step DTM over DTM. One can see that DTM exhibits great error because its graph diverts decreasingly to infinity.

Table 6 shows the maximum differences between the RK4 solution on time steps $h = 0.01$ and $h = 0.001$.

A comparison between two sets of MSDTM solutions and the RK4 solutions is given in Table 7. The first set of result is between a 5 term MSDTM $(h = 0.01)$ and RK4. We could see clearly that the highest error is of $10^{-3}$. Next, we proceed to compare a 5 term MSDTM $(h = 0.001)$ with RK4. The accuracy is strengthened by a maximum error of $10^{-4}$.

**Example 4.4.** Consider the oscillatory equation [21]

$$y'' + \frac{y^3}{1+y^2} = 0, \tag{4.22}$$

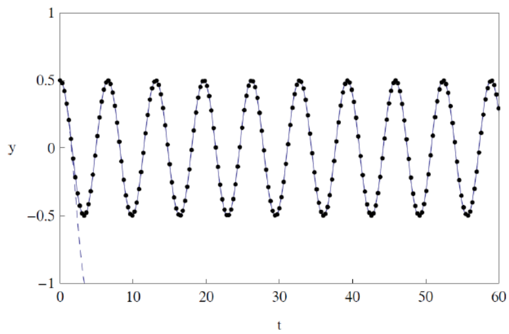



Figure 6: Plots of displacement of $y$ versus time $t$: Dashed line (the DTM solution); Solid line (the multi-step DTM solution, Dotted line (RK4 solution).

Table 6: Absolute differences between RK4 solutions.

| | $\Delta = \lvert RK4_{0.01} - RK4_{0.0001}\rvert$ | | $\Delta = \lvert RK4_{0.001} - RK4_{0.0001}\rvert$ | |
|---|---|---|---|---|
| 10 | 2.8005E-10 | 4.35242E-11 | 2.61041E-14 | 7.27196E-15 |
| 20 | 5.42142E-10 | 1.7748 E-10 | 5.9841E-14 | 2.86993E-14 |
| 30 | 7.6914E-10 | 3.95651E-10 | 9.21763E-14 | 4.45755E-14 |
| 40 | 9.43099E-10 | 6.88071E-10 | 1.09246E-13 | 7.68274E-14 |
| 50 | 1.04535E-9 | 1.04089E-9 | 1.18683E-13 | 1.1624E-13 |
| 60 | 1.0574E-9 | 1.43627E-9 | 1.10745E-13 | 1.59206E-13 |

Table 7: Absolute differences between MSDTM and RK4 solutions.

| | $\Delta = \lvert MSDTM(K=5)_{0.01} - RK4_{0.001}\rvert$ | | $\Delta = \lvert MSDTM(K=5)_{0.001} - RK4_{0.001}\rvert$ | |
|---|---|---|---|---|
| $t$ | $\Delta x$ | $\Delta y$ | $\Delta x$ | $\Delta y$ |
| 10 | 1.58904E-4 | 6.97676E-4 | 1.4238E-5 | 6.96139E-5 |
| 20 | 5.69291E-4 | 1.5793E-3 | 5.40092E-5 | 1.57342E-4 |
| 30 | 1.14969E-3 | 2.80963E-3 | 1.11566E-4 | 2.79816E-4 |
| 40 | 1.76393E-3 | 4.5227E-3 | 1.73732E-4 | 4.5074E-4 |
| 50 | 2.21915E-3 | 6.80578E-3 | 2.21718E-4 | 6.79336E-4 |
| 60 | 2.26893E-3 | 9.68298E-3 | 2.31397E-4 | 9.68766E-4 |

subject to the initial conditions

$$y(0) = 0, \qquad y'(0) = A. \tag{4.23}$$

By using the transformation $y' = x$, we get the following system of differential equations

$$\frac{dy}{dt} = x, \qquad \frac{dx}{dt} = -\frac{y^3}{1+y^2}. \tag{4.24}$$

Also, by this transformation the initial conditions $y(0) = 1$ and $y'(0) = 0$ become

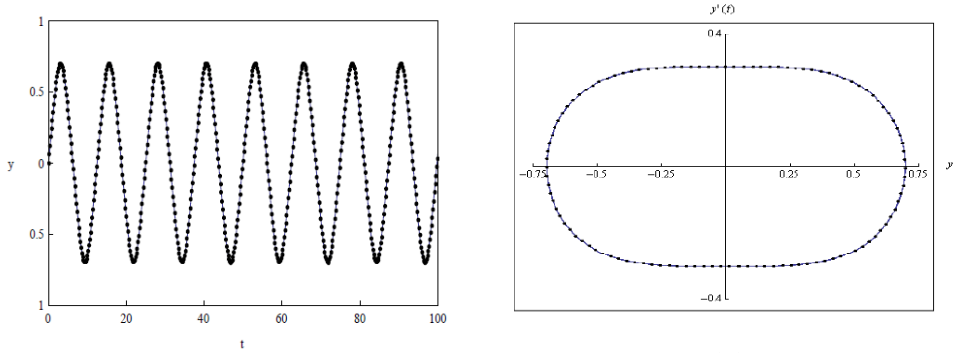
$$y(0) = 0, \qquad x(0) = A. \tag{4.25}$$

Taking the differential transform for Eq. (4.24) with respect to time $t$ gives

$$\begin{cases} Y(k+1) = \dfrac{1}{k+1} X(k), \\ X(k+1) = -\dfrac{1}{k+1} H(k), \end{cases} \tag{4.26}$$

where $X(k)$, $Y(k)$ and $H(k)$ are the differential transforms of the corresponding functions $x(t)$, $y(t)$ and $h(t) = y^3(t)/(1+y^2(t))$, respectively. The initial conditions are given by $Y(0) = 0$ and $X(0) = A$. By using Eq. (2.2), the DTM series solution for the system (4.24) can be obtained as

$$\begin{cases} y(t) = \sum\limits_{n=0}^{N} Y(n) t^n, \\ x(t) = \sum\limits_{n=0}^{N} X(n) t^n. \end{cases} \tag{4.27}$$



(a) (b)

Figure 7: The displacement and phase plane for Example 4.4: Solid (present solution), Dotted line (RK4 solution).

Now, according to the multi-step DTM, taking $N = K \cdot M$, the series solution for the system (4.26) is given by

$$y(t) = \begin{cases} \sum_{n=0}^{K} Y_1(n)t^n, & t \in [0, t_1], \\ \sum_{n=0}^{K} Y_2(n)(t-t_1)^n, & t \in [t_1, t_2], \\ \vdots & \\ \sum_{n=0}^{K} Y_M(n)(t-t_{M-1})^n, & t \in [t_{M-1}, t_M], \end{cases} \tag{4.28a}$$

$$x(t) = \begin{cases} \sum_{n=0}^{K} X_1(n)t^n, & t \in [0, t_1], \\ \sum_{n=0}^{K} X_2(n)(t-t_1)^n, & t \in [t_1, t_2], \\ \vdots & \\ \sum_{n=0}^{K} X_M(n)(t-t_{M-1})^n, & t \in [t_{M-1}, t_M], \end{cases} \tag{4.28b}$$

where $Y_i(n)$ and $X_i(n)$, for $i = 1, 2, \cdots, M$ satisfy the following recurrence relations

$$\begin{cases} Y_i(k+1) = \dfrac{1}{k+1} X_i(k), \\ X_i(k+1) = -\dfrac{1}{k+1} H_i(k), \end{cases} \tag{4.29}$$

such that $Y_i(0) = Y_{i-1}(0)$ and $X_i(0) = X_{i-1}(0)$. By using $Y(0) = 0$, $X(0) = A$ and Eq. (4.29), we can obtain the multi-step DTM solution given in Eqs. (4.28a) and (4.28b). We set the parameter $A = 0.3$ in this example. The values of $M$ and $N$ are taken as 500 and 5000, respectively. Fig. 7 exhibits the comparison between the multi-step

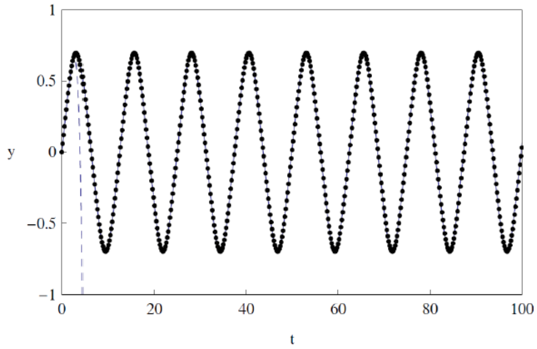



Figure 8: Plots of displacement of $y$ versus time $t$: Dashed line (the DTM solution); Solid line (the multi-step DTM solution, Dotted line (RK4 solution).

DTM solution and the numerical integration results obtained by RK4 method for the displacement and phase diagram of the problem (4.22)-(4.23). From Fig. 7, it is obvious that the solution obtained by the present method is nearly identical with that given by RK4 method.

Table 8: Absolute differences between RK4 solutions.

| | $\Delta = \lvert RK4_{0.01} - RK4_{0.0001}\rvert$ | | $\Delta = \lvert RK4_{0.001} - RK4_{0.0001}\rvert$ | |
|---|---|---|---|---|
| 10 | 3.7876E-11 | 3.2723E-11 | 4.7462E-15 | 1.15463E-14 |
| 20 | 1.90299E-11 | 1.04086E-10 | 4.996E-15 | 8.88178E-16 |
| 30 | 2.48132E-11 | 1.78864E-10 | 2.38698E-15 | 8.27116E-15 |
| 40 | 1.60473E-10 | 9.75143E-11 | 2.17049E-14 | 2.27596E-14 |
| 50 | 1.24906E-12 | 3.03436E-10 | 1.15463E-14 | 8.0113E-14 |
| 60 | 2.32546E-10 | 1.85565E-10 | 7.61613E-14 | 5.00711E-14 |
| 70 | 7.45553E-11 | 3.98835E-10 | 2.10387E-14 | 1.33782E-13 |
| 80 | 6.28648E-11 | 4.86366E-10 | 3.06422E-14 | 1.67755E-13 |
| 90 | 3.85069E-10 | 2.16925E-10 | 1.50172E-13 | 1.00475E-13 |
| 100 | 2.4396E-12 | 6.32865E-10 | 5.38458E-15 | 2.6594E-13 |

Fig. 8 shows the approximate solutions for the problem (4.22)-(4.23) obtained using the multi-step DTM, DTM and RK4 methods. One can see that DTM exhibits great

Table 9: Absolute differences between MSDTM and RK4 solutions.

| | $\Delta = \lvert MSDTM(K=5)_{0.01} - RK4_{0.001}\rvert$ | | $\Delta = \lvert MSDTM(K=5)_{0.001} - RK4_{0.001}\rvert$ | |
|---|---|---|---|---|
| $t$ | $\Delta x$ | $\Delta y$ | $\Delta x$ | $\Delta y$ |
| 10 | 9.27896E-3 | 1.04814E-3 | 4.09822E-4 | 1.06366E-4 |
| 20 | 1.22025E-3 | 2.49172E-2 | 1.46274E-4 | 2.47321E-3 |
| 30 | 1.39416E-2 | 4.33334E-2 | 1.45996E-3 | 4.18997E-3 |
| 40 | 6.02821E-2 | 4.29629E-2 | 5.48484E-3 | 5.01121E-3 |
| 50 | 1.33524E-2 | 1.44777E-1 | 1.3473E-3 | 1.39623E-2 |
| 60 | 1.13715E-1 | 1.20481E-1 | 1.31443E-2 | 8.27576E-3 |
| 70 | 8.8488E-2 | 2.45185E-1 | 3.45222E-3 | 2.68387E-2 |
| 80 | 3.43122E-2 | 3.32062E-1 | 5.94255E-3 | 3.31268E-2 |
| 90 | 1.02038E-1 | 4.79413E-2 | 3.08809E-2 | 1.8273E-2 |
| 100 | 6.1824E-2 | 5.65846E-1 | 2.65661E-3 | 5.60774E-2 |

error because its solution decreasingly goes to infinity.

Table 8 illustrates the maximum differences between the RK4 solutions on time steps considered.

Table 9 shows a comparison between two sets of MSDTM solution and the RK4 solution. The first set of result is between a 5 term MSDTM solution ($h = 0.01$) and RK4 solution. We could see clearly that the highest error is of $10^{-1}$. Next, we proceed to compare a 5 term MSDTM ($h = 0.001$) solution with RK4 solution. The accuracy is strengthened by a maximum error of $10^{-2}$.

## 5 Conclusions

In this study, an algorithm for solving nonlinear oscillators was introduced via MSDTM. Higher accuracy solution was obtained via this algorithm. Comparison between MSDTM solution and RK4 solution is discussed and plotted. The solution via MSDTM is continuous on this domain and analytical at each subdomain.

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