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Collocation Methods for A Class of Volterra Integral Functional Equations with Multiple Proportional Delays

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Abstract. In this paper, we apply the collocation methods to a class of Volterra integral functional equations with multiple proportional delays (VIFEMPDs). We shall present the existence, uniqueness and regularity properties of analytic solutions for this type of equations, and then analyze the convergence orders of the collocation solutions and give corresponding error estimates. The numerical results verify our theoretical analysis.

AMS subject classifications: 65R20, 34K06, 34K28

Key words: Volterra integral functional equation, multiple proportional delays, collocation method.

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1 Introduction

The Volterra integral functional equations with proportional delays (VIFEPDs) provide a powerful model of phenomena when processes are modeled evolving in time, where the rate of change of the process is not only determined by its present state but also by a certain past state. VIFEPDs play an important role in explaining many different phenomena in biology, economy, control theory, electrodynamics, demography, viscoelastic materials and insurance. Numerical methods based on finite difference methods, discontinuous Galerkin methods and spectral methods etc., have also been developed for various VIFEPDs and we refer to [2–5, 8, 9, 11–13, 17], and references therein for details about the rich literature.

In this paper, we shall study the collocation method for Volterra integral functional equations (VIFE) with multiple delay (or: lag) functions $\theta_k = \theta_k(t)$, $k = 1, 2, \cdots, p$ of the form

$$u(t) = \sum_{k=1}^{p} a_k(t)u(\theta_k(t)) + f(t) + (\mathcal{V}u)(t) + \sum_{k=1}^{p}(\mathcal{V}_{\theta_k}u)(t), \quad t \in I := [0, T], \tag{1.1}$$

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where p is some positive integer. The Volterra integral operators $\mathcal{V}$ and $\mathcal{V}_{\theta_k}$ ($k = 1,2,\cdots,p$) are defined by

$$(\mathcal{V}u)(t) := \int_0^t K_0(t,s)u(s)ds, \qquad (\mathcal{V}_{\theta_k}u)(t) := \int_0^{\theta_k(t)} K_k(t,s)u(s)ds,$$

where a_k, f, K_0 and K_k are given smooth functions. The delay functions $\theta_k(t)$, $k = 1,2,\cdots,p$ are assumed to have the following properties:

(P1) $\theta_k(0) = 0$, and θ_k is strictly increasing on I;

(P2) $\theta_k(t) \leq \bar{q}_k t$ on I for some $\bar{q}_k \in (0,1)$;

(P3) $\theta_k \in C^{\nu_k}(I)$ for some integer $\nu_k \geq 0$.

An important special case is the linear vanishing delay or proportional delay, i.e.,

$$\theta_k(t) = q_k t = t - (1-q_k)t := t - \tau_k(t) \quad \text{with } 0 < q_k < 1,$$

which are known as the pantograph delay functions (see [1,7,14,16]). In rest of this paper, we shall concern on the corresponding VIFEMPDs given by

$$u(t) = \sum_{k=1}^{p} a_k(t)u(q_k t) + f(t) + (\mathcal{V}u)(t) + \sum_{k=1}^{p}(\mathcal{V}_{q_k}u)(t), \qquad t \in I, \tag{1.2}$$

where

$$(\mathcal{V}_{q_k}u)(t) := \int_0^{q_k t} K_k(t,s)u(s)ds, \quad k = 1,2,\cdots,p,$$

as the multi-pantograph Volterra integral functional equations.

The collocation method for the Volterra integral equation with proportion delay (VIEPD) of the form

$$u(t) = f(t) + \int_0^t K_0(t,s)u(s)ds + \int_0^{qt} K_1(t,s)u(s)ds, \tag{1.3}$$

with $t \in [0,T]$ is discussed in [6], and recently Hermann and his collaborators also study the collocation method for functional equation

$$u(t) = b(t)u(qt) + f(t), \tag{1.4}$$

where b and f are given functions (see [10]). To the best of our knowledge, there is few work on collocation method for VIFEMPDs of form (1.2). In order to gain some insight approaches for VIFE of first and second kinds, we present a study of piecewise polynomial collocation solutions for (1.2).

There are two main challenges for these VIFEMPDs:

- ◇ the situations for the multiple proportional delays $(\mathcal{V}_{q_k}u)(t)$ in (1.2) are more complicated than single proportional delay;