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The Collocation Method and the Splitting Extrapolation for the First Kind of Boundary Integral Equations on Polygonal Regions

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Abstract. In this paper, the collocation methods are used to solve the boundary integral equations of the first kind on the polygon. By means of Sidi's periodic transformation and domain decomposition, the errors are proved to possess the multi-parameter asymptotic expansion at the interior point with the powers $h_i^3 (i = 1, ..., d)$, which means that the approximations of higher accuracy and a posteriori estimation of the errors can be obtained by splitting extrapolations. Numerical experiments are carried out to show that the methods are very efficient.

AMS subject classifications: 65R10

Key words: Splitting extrapolation; boundary integral equation of the first kind on polygon; collocation method; posteriori estimation.

1 Introduction

By using the single layer potential theory, the plane Dirichlet problem

$$\begin{cases} \triangle u = 0, & (\Omega \quad \text{or} \quad \Omega^c), \\ u = h, & (\Gamma) \end{cases} \tag{1.1}$$

can be converted into a boundary integral equation of the first kind

$$-\frac{1}{\pi} \int_\Gamma g(P) \ln |P - Q| dS_Q = h(P), \quad \forall P \in \Gamma, \tag{1.2}$$

where Ω is a polygon, and Γ is its boundary. The Dirichlet problem on $\Omega^c = R^2/\Omega$ is called an exterior problem.

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We all know that the mathematical theory of the first kind of boundary integral equations is usually more difficult than the second kind due to lack of Fredholm alternative theorem. Although from the viewpoint of the calculation, the work of the discrete matrix generation and the accuracy of the approximation of the first kind of boundary integral equations are better than the second kind, but the mathematical theory of the first kind boundary integral equations is developed only by Sloan and Spence in [1] until 1988. They proved that if the capacity $C_\Gamma \neq 1$, then there was a unique solution in (1.2). Once $g(P)$ was solved, the solution of the interior problem (or exterior problem) can be expressed by

$$u(P) = -\frac{1}{\pi}\int_\Gamma \ln|P-Q|g(Q)dS_Q, \quad \forall P \in R^2\backslash\Gamma. \tag{1.3}$$

Sloan and Spence also used Galerkin method to solve the first kind boundary equations, and proved that using the Galerkin method, the accuracy of the interior-point approximations had superconvergence. However, the computational complexity of Galerkin method was too huge. Yan and other authors in [2] used the constant element collocation method to solve (1.2) and got the error estimate at the interior point with $O(h^{\beta+3/2})$, where $\beta = (1-\alpha)/\alpha$ and $\alpha\pi$ were the largest interior angle of Γ. This means that the accuracy reduces on concave regions. Thus, Yan in [3] recommended getting the high accuracy by mesh grading, which undoubtedly increased the difficulty of calculating. By using the mechanical quadrature method Lu Tao and Huang Jin in [4] proved the convergence of approximate solutions and the asymptotic expansions of the error, which can be used to accelerate the convergence by Richardson's extrapolation.

Splitting extrapolation method (SEM) based on a multivariate asymptotic expansion of the error is an effective parallel algorithm, which possesses high order of accuracy and high degree of parallelism (see [6]). By means of SEM, a large problem can be turned into many smaller discrete problems involving several grid parameters. If the errors of approximations of the problems have the multivariate asymptotic expansions, then after solving these small subproblems in parallel, the higher accuracy is computed by SEM.

In this paper, the collocation methods are used to solve the boundary integral equations of the first kind on the polygons. By means of Sidi's periodic transformation (see [5]) and domain decomposition, the errors are proved to possess the multi-parameter asymptotic expansion at the interior point with the powers $h_i^3 (i = 1, \cdots, d)$, which means that the approximations of higher accuracy and a posteriori estimation of the errors can be got by SEM.

In section 2, we will discuss the collocation method for the first kind of boundary integral equations on a circle. It will show that the error at the interior point have the asymptotic expansion. Based on section 2, further analysis for solving the first kind of boundary integral equations on a polygonal domain will be carried out. In section 3, using the results of the circle and the midpoint trapezoidal formula, the multi-parameter asymptotic expansion of the error at the interior point with the powers $h_i^3(i=1,\cdots,d)$ will be obtained, which means that by using the splitting extrapolation, high accuracy order $O(h^5)$ can be proposed. Some examples will be shown in section 4.

2 Collocation on a circle

In this section, let Ω be a circle with radius $e^{-1/2}$, where the boundary Γ is described by a 2π-periodic function $\gamma(s)=(\gamma_1(s),\gamma_2(s))$ with $|\gamma'(s)|=[(\gamma_1'(s))^2+(\gamma_2'(s))^2]^{1/2}>0$. Then Eq. (1.2) can be written as

$$A\omega=f, \tag{2.1}$$

where

$$A\omega=\int_{-\pi}^{\pi}\Lambda(s-\sigma)\omega(\sigma)d\sigma, \tag{2.2a}$$

$$\Lambda(s-\sigma)=-\frac{1}{\pi}\ln|2e^{\frac{-1}{2}}\sin\frac{s-\sigma}{2}|. \tag{2.2b}$$

To derive the collocation equation for (2.1)-(2.2), we take the step $h=2\pi/N$, and set

$$\sigma_j=-\pi+hj,\qquad \text{for}\quad j=0,1,\cdots,N,$$
$$\sigma_{j+\frac{1}{2}}=\sigma_j+\frac{1}{2}h,\quad \text{for}\quad j=0,1,\cdots,N-1.$$

Then, divide the interval $[-\pi,\pi]$ uniformly. Suppose that S^h is a piecewise constant function space with break points $\{\sigma_j\}_{j=0}^N$ and Q^h is an interpolation projection defined by

$$Q^hv=\sum_{j=0}^{N-1}v(\sigma_{j+\frac{1}{2}})X_j(s),$$

where

$$\begin{cases} X_j(s)=1, & s\in[\sigma_j,\sigma_{j+1}],\\ X_j(s)=0, & s\notin[\sigma_j,\sigma_{j+1}].\end{cases}$$

Then the collocation equation of (2.1) is to find $w^h\in S^h$ satisfying

$$A\omega^h|_{\sigma_{j+\frac{1}{2}}}=f(\sigma_{j+\frac{1}{2}}),\quad j=0,\cdots,N-1. \tag{2.3}$$

Once $\{\omega^h(\sigma_{j+1/2}),j=0,\cdots,N-1\}$ is solved, from (1.3) the approximation solution $u^h(z)$ of $u(z)$ can be derived by

$$u^h(z)=-\frac{h}{\pi}\sum_{j=0}^{N-1}\ln|z-\gamma(\sigma_{j+\frac{1}{2}})|\omega^h(\sigma_{j+\frac{1}{2}}),\quad \forall z\in\Omega.$$

The collocation equation (2.3), in operator form, is expressed as follows

$$A_h\omega^h = Q^h f, \tag{2.4}$$

where $A_h = Q^hAQ^h$. After ω is solved in (2.1), then

$$u(z) =< \omega, T_z >= \int_{-\pi}^{\pi} \ln|\gamma(s) - z|\omega(s)ds, \quad z \in \Omega.$$

Since $z \notin \Gamma$, we have $T_z = \ln|\gamma(s) - z|$ is smooth. Moreover, once ω^h is solved in (2.4), then

$$u^h(z) =< \omega^h, T_z >,$$

where

$$< f, g >= \int_{-\pi}^{\pi} f(s)\overline{g}ds.$$

Below we assume that $H^r(2\pi)$ is a Sobolev space with 2π-periodic functions. The following lemmas can be seen in [2].

Lemma 2.1. *If $f \in H^r$, then f has a Fourier expansion*

$$f(s) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} \hat{f}_j e^{ijs}, \qquad i = \sqrt{-1},$$

where $\hat{f}_j = 1/\sqrt{2\pi}\int_{-\pi}^{\pi} f(s)e^{-ijs}ds$ and $\exists c > 0$, such that $|\hat{f}_j| \le c/|j|^r$.

Lemma 2.2. *The eigenvalues of operator A are given by*

$$\begin{cases} \mu_j = 1, & j = 0, \\ \mu_j = \frac{1}{|j|}, & j \neq 0, \end{cases}$$

and the corresponding eigenfunctions are $e^{\pm ijs}$.

Lemma 2.3. *The eigenvalues of the collocation operator $A_h = Q^hAQ^h : S^h \to S^h$ are given by*

$$\begin{cases} \lambda_p = 1, & p = 0, \\ \lambda_p = \frac{N}{\pi}\sin\frac{\pi|p|}{N}\sum\limits_{k=0}^{\infty}(-1)^k\big(\frac{1}{(kN+|p|)^2} + \frac{1}{(kN+N-|p|)^2}\big), & p \in \Lambda_h^*, \end{cases}$$

and the corresponding eigenfunctions are

$$e_h^p(s) = \sum_{j=0}^{N-1} e^{ihjp}X_j(s), \quad p \in \Lambda_h,$$

where $\Lambda_h = \{p : |p| \le (N-1)/2\}, \Lambda_h^* = \Lambda_h \setminus \{0\}$. Moreover, $\{e_h^p, p \in \Lambda_h\}$ constructs an orthogonal basis of S^h satisfying

$$< e_h^p, e_h^{p'} >= 2\pi\delta_{pp'}, \quad p, p' \in \Lambda_h. \tag{2.5}$$

Since each 2π-periodic function $f(s)$ has a Fourier expansion $f(s) = \sum_{j=-\infty}^{\infty} \hat{f}_j e^{ijs}$, we can let

$$P_N f = \sum_{j\in\Lambda_h} \hat{f}_j e^{ijs}.$$

Obviously P_N is a projection operator on span $\{e^{ijs}, j \in \Lambda_h\}$.

Lemma 2.4. *Let $Q_N = I - P_N, \forall u \in H^r$. Then*

$$\| Q_N u \|_t \leq ch^{r-t},$$

where $\| \bullet \|$ is the norm of H^t .

Lemma 2.5. *It holds that*

$$P^h e^{ims} = \alpha_m e_h^m, \qquad \alpha_m = 2(mh)^{-1} \sin\big(\frac{mh}{2}\big) e^{im\sigma_{\frac{1}{2}}}, \tag{2.6a}$$

$$Q^h e^{ims} = \beta_m e_h^m, \quad \beta_m = e^{im\sigma_{\frac{1}{2}}}. \tag{2.6b}$$

where P^h is the orthogonal projection operator to S^h.

Since the collocation solution $\omega^h = A_h^{-1} Q^h f = A_h^{-1} Q^h A\omega$, we can define an operator

$$G_h = A_h^{-1} Q^h A,$$

so that $\omega^h = G_h\omega$. Let $e_p = e^{ips}$, and $H_N = \{e_j, j \in \Lambda_h\}$. Obviously we have

$$P_N G_h : H_N \longrightarrow H_N.$$

Now we prove the following lemma

Lemma 2.6. *It holds that*

$$P_N e_h^p = \overline{\alpha}_p e^{ips} = 2(ph)^{-1} \sin\big(\frac{ph}{2}\big) e^{-im\sigma_{\frac{1}{2}}} e^{ips}. \tag{2.7}$$

Proof. Since e_h^p has a Fourier expansion

$$e_h^p = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \int_{-\pi}^{\pi} e_h^p(s) e^{-ijs} ds e^{ijs},$$

we have

$$\begin{aligned} P_N e_h^p =& \frac{1}{2\pi} \sum_{j\in\Lambda_h} \int_{-\pi}^{\pi} e_h^p(s) e^{-ijs} ds e^{ijs} = \frac{1}{2\pi} \sum_{j\in\Lambda_h} < e_h^p, e^{ijs} > e^{ijs} \\ =& \frac{1}{2\pi} \sum_{j\in\Lambda_h} < P^h e_h^p, e^{ijs} > e^{ijs} = \frac{1}{2\pi} \sum_{j\in\Lambda_h} < e_h^p, P^h e^{ijs} > e^{ijs} \\ =& \frac{1}{2\pi} \sum_{j\in\Lambda_h} \overline{\alpha}_j < e_h^p, \alpha_j e^{ijs} > e^{ijs} = \sum_{j\in\Lambda_h} \overline{\alpha}_j \delta_{jp} e^{ijs} = \overline{\alpha}_p e^{ips}, \end{aligned}$$

where we have not only used $P_N e_h^p = e_h^p$ and the self-conjugate properties of P^h, but also used Eqs. (2.5) and (2.6).

Theorem 2.1. *The following result holds:*

$$\begin{cases} P_N G_h e_p = \dfrac{2\frac{1}{ph}\sin\frac{ph}{2}}{|p|\lambda_p} e_p, & p \neq 0, \\ P_N G_h e_p = 1, & p = 0. \end{cases}$$

Proof. Since $p = 0$ is easy, we assume that $p \neq 0$, then

$$\begin{aligned} P_N G_h e_p &= P_N A_h^{-1} Q^h A e_p = \frac{1}{|p|} P_N A_h^{-1} Q^h e_p \\ =&\frac{\beta_p}{|p|} P_N A_h^{-1} e_h^p = \frac{\beta_p}{|p|\lambda_p} P_N e_h^p = \frac{\beta_p \overline{\alpha}_p}{|p|\lambda_p} e^{ips} = \frac{2\frac{1}{ph}\sin\frac{ph}{2}}{|p|\lambda_p} e^{ips}. \end{aligned}$$

This completes the proof.

Theorem 2.2. *If $p \neq 0$, the eigenvalue λ_p has the following asymptotic expansion*

$$\lambda_p = \frac{N}{\pi} \sin\frac{\pi|p|}{N} \Big(\frac{1}{p^2} - \frac{4|p|}{N^3}\eta(0) + O(\frac{|p|^3}{N^5}) \Big), \tag{2.8}$$

where

$$\eta(x) = \sum_{k=1}^{\infty} (-1)^k \frac{k}{(k^2 - x)^2}.$$

Proof. It follows from Lemma 2.3 that if $p \neq 0$, the eigenvalue λ_p can be written as

$$\lambda_p = \frac{N}{\pi} \sin\frac{\pi|p|}{N} \sum_{k=0}^{\infty} (-1)^k \Big(\frac{1}{(kN+|p|)^2} + \frac{1}{(kN+N-|p|)^2} \Big). \tag{2.9}$$

Consequently,

$$\begin{aligned} &\sum_{k=0}^{\infty} (-1)^k \Big(\frac{1}{(kN+|p|)^2} + \frac{1}{(kN+N-|p|)^2} \Big) \\ =&\frac{1}{|p|^2} + \sum_{k=1}^{\infty} (-1)^k \Big(\frac{1}{(kN+|p|)^2} - \frac{1}{(kN-|p|)^2} \Big) \\ =&\frac{1}{|p|^2} - 4|p|N \sum_{k=1}^{\infty} (-1)^k \frac{k}{N^4 (k^2 - \frac{p^2}{N^2})^2} \\ =&\frac{1}{|p|^2} - \frac{4|p|}{N^3} \eta(\frac{p^2}{N^2}), \end{aligned}$$

where

$$\eta(x)=\sum_{k=1}^{\infty}(-1)^k\frac{k}{(k^2-x)^2},\quad x\in(0,1).$$

Since $\eta(x)$ is analytical, its Taylor expansion is

$$\eta(x)=\eta(0)+\eta'(0)x+\cdots+\frac{\eta^m(0)}{m!}+\cdots,$$

which yields

$$\sum_{k=0}^{\infty}(-1)^k\Big(\frac{1}{(kN+|p|)^2}+\frac{1}{(kN+N-|p|)^2}\Big)=\frac{1}{p^2}-\frac{4|p|}{N^3}\eta(0)+O(\frac{|p|^3}{N^5}). \tag{2.10}$$

Substituting (2.10) into (2.9), the proof of Theorem 2.2 is complete.

Theorem 2.3. *For $f\in H^r, r>3$, we have*

$$u(z)-u^h(z)=T_z(\omega-\omega^h)=Ch^3+O(h^5),$$

where z can be any value outside Γ, and $C\in R$.

Proof. Substituting (2.8) into (2.7), we obtain

$$\begin{aligned}P_NG_he_p&=2e_p\frac{\sin\frac{ph}{2}}{|p|h}\Big(|p|\frac{N}{\pi}\sin\frac{\pi|p|}{N}\sum_{k=0}^{\infty}(-1)^k(\frac{1}{(kN+|p|)^2}+\frac{1}{(kN+N-|p|)^2})\Big)^{-1}\\&=e_p\Big(p^2(\frac{1}{p^2}-\frac{4|p|}{N^3}\eta(0)+O(\frac{|p|^3}{N^5}))\Big)^{-1}=e_p\Big(1-\frac{4|p|^3}{N^3}\eta(0)+O(\frac{|p|^3}{N^5})\Big)^{-1}\\&=\Big(1+\frac{4|p|^3}{N^3}\eta(0)+O(\frac{|p|^3}{N^5})\Big)e_p,\qquad\forall p\in\Lambda^*.\end{aligned}\tag{2.11}$$

Since

$$e_0=1,\quad P_NG_he_0=e_0,\quad \omega(s)=\sum_{j=-\infty}^{\infty}\hat{\omega}_je_j,\quad e_j=e^{ijs},$$

using (2.11) we gives

$$P_NG_h\omega=P_N\omega^h=\sum_{p\in\Lambda_h}\Big(1+\frac{4|p|^3}{N^3}\eta(0)+O(\frac{|p|^3}{N^5})\Big)P_N\omega. \tag{2.12}$$

Notice that T_z is smooth, then we have

$$\begin{aligned}T_z(\omega-G_h\omega)&=<\omega-\omega^h,T_z>=<\omega-G_h\omega,T_z>\\&=T_z(P_N(\omega-G_h\omega))+T_z\big(Q_N(\omega-G_h\omega)\big)=I_1+I_2.\end{aligned}$$

Since $T_z\in H^5$, from Lemma 2.4 we know that the accuracy of I_2 is $O(1/N^5)$.

Using (2.12) and some simple calculations ,we get

$$P_N\omega - P_N G_h \omega = -\sum_{p\in\Lambda_h} \frac{4|p|^3}{N^3}\eta(0)\overset{\wedge}{\omega}_p e_p.$$

Therefore,

$$T_z(P_N\omega - P_N G_h\omega) = \frac{4\sqrt{2\pi}\eta(0)}{N^3}\sum_{p\in\Lambda_h}\overset{\wedge}{\omega}_p|p|^3\overset{\wedge}{T_z^p}$$
$$=\frac{4\sqrt{2\pi}\eta(0)}{N^3}\sum_{p=-\infty}^{\infty}\overset{\wedge}{\omega}_p|p|^3\overset{\wedge}{T_z^p}+O(\frac{1}{N^5}).$$

Let

$$-\sqrt{2\pi}\sum_{p\in\Lambda_h}\overset{\wedge}{\omega}_p|p|^3\overset{\wedge}{T_z^p} =< \omega, A^{-3}T_z >,$$

where

$$A^{-3}f = \sum_{j=-\infty}^{\infty}\frac{\overset{\wedge}{f_j}}{|j|^{-3}}e^{ijs}.$$

Since $f \in H^r, r > 3$, the error at the interior point is

$$T_z(\omega - \omega^h) = \frac{-4\eta(0)}{N^3} < \omega, S > +O(\frac{1}{N^5}),$$

where $S = A^{-3}T_z$ is independent of h. Therefore the approximation solution $u^h(z)$ has the asymptotic expansion with h^3 power. The proof is complete.

3 Collocation on a polygon

Let $\Gamma = \cup_{j=1}^{d}\overline{\Gamma}_j$ be a domain decomposition for the boundary Γ of a polygon Ω, where $\Gamma_j, j = 1,\cdots,d$ are smooth arcs and $T_j, j = 1,\cdots,d$ be corner points with the interior angle θ_j . We allow that $\theta_j = \pi$, if T_j is a smooth point of Γ_j. Assume that Γ_j can be described by parameter form

$$z_j(s) = \big(x_j(s), y_j(s)\big) : [0,1] \to \Gamma_j,$$
$$|z_j'(s)| = [(x_j'(s))^2 + (y_j'(s))^2]^{1/2} > 0, \quad j = 1,\cdots,d.$$

Then under the change of variables Eq. (1.2) becomes the following boundary integral equation system of the first kind

$$\sum_{j=1}^{d} -\frac{1}{\pi}\int_0^1 \ln|z_i(t) - z_j(s)||z_j'(s)|v_j(s)ds = f_i(t), \quad i = 1,\cdots,d, \tag{3.1}$$

where $v_j(s) = g(z_j(s)), f_i(t) = h(z_i(t))$. Since $v_j(s)$ is singular at the integral endpoints, we should eliminate the singularity to improve the accuracy of the approximation solution. For example, we can use Sidi's Sin^l-transformation

$$\psi_p(t) = \theta_p(t)/\theta_p(1), \quad \theta_p(t) = \int_0^t (sin(\pi s))^p ds,$$

to eliminate the singularity, because its derivative $\psi_p'(t)$ has p-order zero point at $t = 0$ and $t = 1$. For example,

$$\psi_1(t) = \frac{1}{2}\Big(1 - cos(\pi t)\Big),$$

and its derivative $\psi_1'(t) = \pi/2sin(\pi t)$ with one order zero point at $t = 0$ and $t = 1$. Hence, under the transformation $\psi_p(t)$, (3.1) becomes

$$Kw = F, \tag{3.2}$$

where $K = [K_{ij}]_{i,j=1}^d$ is an integral operator matrix, and the kernel of the operator K_{ij} is

$$k_{ij}(t,s) = -\frac{1}{\pi} \ln |z_i(\psi_p(t)) - z_j(\psi_p(s))|, \quad 0 \le t, s \le 1.$$

Let $w = (w_1, \cdots, w_d)^T$ with

$$w_j(s) = v_j(\psi_p(s))|z'(\psi_p(s))|\psi_p'(s), \quad j = 1, \cdots, d$$

be the unknown vector function of (3.2), and $F = (F_1, \cdots, F_d)^T$ with $F_j(t) = f_j(\psi_p(t))$ be the functions of the right hand side of (3.2).

Now we take steps $h_j = 1/N_j$ on $\Gamma_j (j = 1, \cdots, d)$, and set $t_{j,i} = ih_j,\ i = 1, \cdots, N_j$, which constructs a subdivision on Γ_j

$$\Lambda_j : 0 = t_{j0} < t_{j1} < \cdots < t_{j,N_j} = 1.$$

Let S_j^h be a piecewise constant function subspace under Λ_j and $t_{j,i-1/2} = (i-1/2)h_j, i = 1, \cdots, N_j$, i.e., The mid-point $t_{j,i+1/2}$ of the interval $[t_{ji}, t_{j,i+1}]$ are interpolation nodes. Thus the collocation equation responding to the boundary integral equation (3.2) is to find $\omega_j^h(s) \in S_j^h, j = 1, \cdots, d$ satisfying

$$I^h K \omega^h = I^h F, \tag{3.3}$$

where

$$w^h = (w_1^h, \cdots, w_d^h)^T \in (S_j^h)^d,$$

and $I^h : (C[0,1])^d \to (S_j^h)^d$ is an interpolation operator. Obviously (3.3) can be written as

$$\sum_{j=1}^d \sum_{i=1}^{N_j} \int_{t_{ji}}^{t_{j,i+1}} k_{ij}(t_{i,m+\frac{1}{2}}, s)\omega_j^h(s)ds = F_i(t_{i,m+\frac{1}{2}}), \quad i = 1, \cdots, d;\ m = 1, \cdots, N_j - 1,$$

or

$$\sum_{j=1}^{d}\sum_{i=1}^{N_j}\omega_{j,i-\frac{1}{2}}\int_{t_{ji}}^{t_{j,i+1}}k_{ij}(t_{i,m+\frac{1}{2}},s)ds=F_i(t_{i,m+\frac{1}{2}}),\quad i=1,\cdots,d;\ m=1,\cdots,N_j-1,$$

where $\omega_{j,i-\frac{1}{2}}=\omega_j^h(t_{j,i-\frac{1}{2}})$. Let

$$\widetilde{K}_{ij,mr}=\sum_{j=1}^{d}\sum_{r=1}^{N_j}\int_{t_{jr}}^{t_{j,r+1}}k_{ij}(t_{i,m+\frac{1}{2}},s)ds.$$

Then the constant element collocation method becomes to solve the linear equations

$$\widetilde{K}\widetilde{w}=\widetilde{F}, \tag{3.4}$$

where $\widetilde{K}=[\widetilde{K}_{ij}]_{i,j=1}^{d}$ is a matrix with block structure, and the block $\widetilde{K}_{ij}$ is of size N_i by N_j . Moreover

$$\widetilde{F}=\Big(F_1(t_{1,1+\frac{1}{2}}),\cdots,F_1(t_{1,N_1+\frac{1}{2}}),\cdots,F_d(t_{1,1+\frac{1}{2}}),\cdots,F_d(t_{1,N_d+\frac{1}{2}})\Big)^T,$$
$$\widetilde{w}=\Big(w_{1,1+\frac{1}{2}},\cdots,w_{1,N_1+\frac{1}{2}},\cdots,w_{d,1+\frac{1}{2}},\cdots,w_{d,N_d+\frac{1}{2}}\Big)^T.$$

Once (3.4) is solved, from (1.3) the approximation solution of the interior problem (or exterior problem) at the point $P(\overline{x},\overline{y})\in R^2\backslash\Gamma$ can be obtained by

$$u^h(P)=-\frac{1}{2\pi}\sum_{j=1}^{d}\sum_{i=1}^{N_j}w_{j,i+\frac{1}{2}}\int_{t_{ji}}^{t_{j,i+1}}\ln\Big[\Big(\overline{x}-x_j(\psi_p(t))\Big)^2+\Big(\overline{y}-y_j(\psi_p(t))\Big)^2\Big]dt. \tag{3.5}$$

For further analysis we consider the integral equations (3.2). Decompose $K_{ii}=A_{ii}+B_{ii}$, where

$$a(t,s)=-\frac{1}{\pi}\ln\Big|2e^{\frac{-1}{2}}\sin\big(\pi(\psi_p(t)-\psi_p(s))\big)\Big|,$$

is the kernel of A_{ii} and $b_i(t,s)=k_{ii}(t,s)-a(t,s)$ is the kernel of B_{ii}. Obviously

$$\begin{cases} b_i(t,s)=-\frac{1}{\pi}\ln\Big|\frac{z_i(\psi_p(t))-z_i(\psi_p(s))}{2e^{\frac{-1}{2}}\sin(\pi(\psi_p(t)-\psi_p(s)))}\Big|, & t-s\neq 0,\\ b_i(t,s)=-\frac{1}{\pi}\ln|e^{\frac{1}{2}}z_i(\psi_p(t))|, & t-s=0. \end{cases}$$

If we let $A=diag(A_{11},\cdots,A_{dd})$, then the operator K can be decomposed into $K=A+B$.

Consider the discrete equation (3.4) of the integral equations (3.2). Then the matrix $\widetilde{K}$ can be decomposed as $\widetilde{K}=\widetilde{A}+\widetilde{B}$,where

$$\widetilde{A}=diag(\widetilde{A}_{11},\cdots,\widetilde{A}_{dd}),$$
$$\widetilde{A}_{ii}=-\frac{1}{\pi}cirle\Big(h_i\ln\big(\frac{h_i}{\pi e^{\frac{1}{2}}}\big),\ h_i\ln\big(2e^{\frac{-1}{2}}\sin(\pi h_i)\big),\cdots,$$
$$h_i\ln\big(2e^{\frac{-1}{2}}\sin((N_i-1)\pi h_i)\big)\Big),\qquad i=1,\cdots,d$$

are circulant matrices.

Using the results of Section 2 and [3] we derive the following lemma.

Lemma 3.1. *It holds that*

1. *The kernel $b_i(t,s)$ is a smooth function in $[0,1]^2$, and its derivative of any order exists;*
2. *$B_{ii}: H^t \to H^{t+\gamma},\ t \in R,\ \gamma \geq 1$ is a bounded operator;*
3. *$A_{ii}^{-1}B_{ii}$ is a compact operator from H^t to H^t;*
4. *If $C_\Gamma \neq 1$, then $K_{ii}: H^t \to H^{t+1},\ t \in R$ is a bijection, and we have*

$$\widetilde{A}_{ii}\omega^{h_i} + \widetilde{B}_{ii}\omega^{h_i} = F_i, \quad \widetilde{A}_{ii}\omega^{h_i} - A_{ii}\omega = Ch_i^3 + O(h_i^5),$$

where $C \in R$ is a constant.

The following lemma can be found in [6].

Lemma 3.2. *If $g \in C^{2k+1}[a,b]$, then*

$$\begin{aligned} M_n(f) - \int_a^b g(x)dx = \frac{C_2}{2!}h^2[g'(b) - g'(a)] + \frac{C_4}{4!}h^4[g'''(b) - g'''(a)] + \cdots \\ + \frac{C_{2k}}{(2k)!}h^{2k}[g^{2k-1}(b) - g^{2k-1}(a)] + O(h^{2k+1}), \end{aligned}$$

where $f(x) = (b-a)g\big(a + (b-a)x\big)$, $x \in [0,1]$, and $M_n(f)$ is the midpoint trapezoidal formula of f,

$$C_{2j} = B_{2j}\Big(\frac{1}{2}\Big) = -(1 - 2^{1-2j})B_{2j}, \quad j = 1, \cdots, k,$$

and B_{2j} is the Bernoulli polynomial.

Corollary 3.1. *For any $\widetilde{B}_{ij} \in H^r$, we have*

$$\widetilde{B}_{ij}\omega^{h_j} - B_{ij}\omega = O(h_j^{2r}).$$

Using Theorem 2.3 and Corollary 3.1, we can directly obtain the following theorem.

Theorem 3.1. *Suppose that Ω is a polygon with piecewise smooth boundary Γ, and the capacity $C_\Gamma \neq 1$. Then there exits a function $\Phi = (\phi_1, \cdots, \phi_d)^T$, independent of $h = (h_1, \cdots, h_d)$, such that the approximation solution derived from* (3.5) *has the following multi-parameter asymptotic expansion*

$$u^h(P) - u(P) = diag(h_1^3, \cdots, h_d^3)\Phi + O(h_0^5), \tag{3.6}$$

where $h_0 = \max_{i=1,\cdots,d} h_i$.

Under the asymptotic expansion (3.6), the splitting extrapolation can be carried out by the following algorithm.

Algorithm (SEM):

Step 1 Let $h^{(0)}=(h_1,\cdots,h_d)$, $h^{(i)}=(h_1,\cdots,h_{i/2},\cdots,h_d)$, then solve the equations (3.4) to get the solutions $u_j^{(i)}(t_{jm}), j=1,\cdots,d; m=1,\cdots,N_j$ parallelly;

Step 2 Calculate the value of the splitting extrapolation u^* on the coarse grid point by the formula

$$u_j^*(t_{jm})=\frac{8}{7}\Big[\sum_{i=1}^{d}u_j^{(i)}(t_{jm})-\big(d-\frac{7}{8}\big)u_j^{(0)}(t_{jm})\Big],\quad j=1,\cdots,d;\ m=1,\cdots,N_j; \tag{3.7}$$

Step 3 From (3.7) we can easily derive an important asymptotic posterior estimation

$$\begin{aligned}&|u_j(t_{jm})-\frac{1}{d}\sum_{i=1}^{d}u_j^{(i)}(t_{jm})|\\ \le&\Big|u(t_{jm})-\frac{8}{7}\Big[\sum_{i=1}^{d}u_j^{(i)}(t_{jm})-(d-\frac{7}{8})u_j^{(0)}(t_{jm})\Big]\Big|+(\frac{8}{7}d-1)\Big|\frac{1}{d}\sum_{i=1}^{d}u_j^{(i)}(t_{jm})-u_j^{(0)}(t_{jm})\Big|\\ \le&(\frac{8}{7}d-1)\Big|\frac{1}{d}\sum_{i=1}^{d}u_j^{(i)}(t_{jm})-u_j^{(0)}(t_{jm})\Big|+O(h_0^5),\end{aligned} \tag{3.8}$$

which can be used to check the accuracy promptly in the process of the actual calculation.

4 Numerical experiments

In this section we shall give two numerical experiments to show the methods in this paper are very efficient.

Example 4.1. Consider the equation (1.1), here $\Omega=(-1,1)^2\setminus(0,1)\times(0,-1)$. Take the exact solution as $u(r,\theta)=r^{3/2}\cos(3\theta/2)$ under the polar coordinate, i.e., the origin point is a singular point of the boundary integral equation. Using SEM, The error at the interior point $(-0.5,-0.5)$ is as follows

Table 1: The error at the interior point (-0.5,-0.5).

$N=(N_1,N_2,N_3,N_4,N_5,N_6)$	$u(-0.5,-0.5)-u^h(-0.5,-0.5)$
(8,8,8,8,8,8)	3.5133 e-05
(16,8,8,8,8,8)	3.9814e-05
(8,16,8,8,8,8)	3.5308e-05
(8,8,16,8,8,8)	5.6309e-06
(8,8,8,16,8,8)	1.001e-04
(8,8,8,8,16,8)	4.5369e-05
(8,8,8,8,8,16)	5.9794e-05
posteriori error estimation	5.7433e-06
splitting extrapolation error	4.25e-06

Example 4.2. Consider the equation (1.1),

$$\Omega = (0,1)^2, \quad \Omega^c = R^2 \backslash (0,1)^2, \quad u|_\Gamma = x^2 + y^2.$$

Obviously the interior problem is on a convex region, however, the exterior problem is on a concave region, where the singularity at the concave point is $(s-s_0)^{-1/3}$. Using Galerkin method, the approximate solution at $(0.6, 0.6)$ for the interior problem and the approximate solution at $(1.2, 1.2)$ for the exterior problem are given in [1], and some numerical results have been shown in Table 2. Using the collocation method in this paper, the results of the splitting extrapolation method and the posteriori error estimation are shown in Table 3 under transformation $\psi_1(t)$ and Table 4 under transformation $\psi_3(t)$.

Table 2: Numerical results in [1].

N	$u(1.2,1.2) - u^h(1.2,1.2)$	$u(0.6,0.6) - u^h(0.6,0.6)$
16	2.13e-02	1.0657e-02
32	7.23e-03	1.599e-03
64	2.61e-03	2.33e-04
128	1.01e-03	1.8e-05
256	4.2e-04	7.0e-06
512	1.9e-04	1.22e-06
Exact solution	0.6122	0.994977

Table 2 shows that the convergence speed is slow for Galerkin method. Tables 3 and 4 show that using SEM and Sidi's Sin^l-transformation, the collocation method in this paper is with high accuracy and low computational complexity.

Table 3: The error at (0.6,0.6) for interior problem.

$N = (N_1, N_2, N_3, N_4)$	$u(0.6,0.6) - u^h(0.6,0.6)$
(8,8,8,8)	2.42e-04
(16,8,8,8)	1.7179e-04
(8,16,8,8)	2.1404e-04
(8,8,16,8)	2.1404e-04
(8,8,8,16)	1.7179e-04
posteriori error estimation	4.9281e-05
splitting extrapolation error	1.6917e-05

Table 4: The error at (1.2,1.2) for exterior problem.

$N = (N_1, N_2, N_3, N_4)$	$u(1.2,1.2) - u^h(1.2,1.2)$
(8,8,8,8)	9.0355e-04
(16,8,8,8)	9.1626e-04
(8,16,8,8)	4.5293e-04
(8,8,16,8)	4.5293e-04
(8,8,8,16)	9.1626e-04
posteriori error estimation	2.1896e-04
splitting extrapolation error	9.7394e-05

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