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Assessing the Performance of a Three Dimensional Hybrid Central-WENO Finite Difference Scheme with Computation of a Sonic Injector in Supersonic Cross Flow

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Abstract. A hybridization of a high order WENO-Z finite difference scheme and a high order central finite difference method for computation of the two-dimensional Euler equations first presented in [B. Costa and W. S. Don, J. Comput. Appl. Math., 204(2) (2007)] is extended to three-dimensions and for parallel computation. The Hybrid scheme switches dynamically from a WENO-Z scheme to a central scheme at any grid location and time instance if the flow is sufficiently smooth and vice versa if the flow is exhibiting sharp shock-type phenomena. The smoothness of the flow is determined by a high order multi-resolution analysis. The method is tested on a benchmark sonic flow injection in supersonic cross flow. Increase of the order of the method reduces the numerical dissipation of the underlying schemes, which is shown to improve the resolution of small dynamic vortical scales. Shocks are captured sharply in an essentially non-oscillatory manner via the high order shock-capturing WENO-Z scheme. Computations of the injector flow with a WENO-Z scheme only and with the Hybrid scheme are in very close agreement. Thirty percent of grid points require a computationally expensive WENO-Z scheme for high-resolution capturing of shocks, whereas the remainder of grid points may be solved with the computationally more affordable central scheme. The computational cost of the Hybrid scheme can be up to a factor of one and a half lower as compared to computations with a WENO-Z scheme only for the sonic injector benchmark.

AMS subject classifications: 65P30, 77Axx

Key words: Weighted essentially non-oscillatory, central difference, multi-resolution, hybrid, shock, injector.

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1 Introduction

Conservative Weighted Essentially Non-Oscillatory finite difference schemes (WENO) have been developed in recent years as a class of high order (resolution) method for solutions of hyperbolic conservation laws (PDEs) in the presence of shocks and small scale structures in the solution (for a detail review of WENO schemes, see [1] and references contained therein). WENO schemes owe their success to the use of a dynamic set of substencils where a nonlinear convex combination of lower order polynomials *adapts* either to a higher order polynomial approximation at smooth parts of the solution, or to a lower order polynomial approximation that avoids interpolation across discontinuities. The upwinding of the spatial discretization provides the necessary dissipation for shock capturing.

The local computational stencils of $(2r-1)$ order WENO schemes are composed of r overlapping substencils of r points, forming a larger stencil with $(2r-1)$ points. The scheme yields a local rate of convergence that goes from order r at the non-smooth parts of the solution, to order $(2r-1)$ when the convex combination of local lower order polynomials is applied at smooth parts of the solution. The nonlinear coefficients of WENO's convex combination, hereafter referred to as *nonlinear weights* ω_k, are based on lower order local smoothness indicators β_k, $k=0,\cdots,r-1$ that measure the sum of the normalized squares of the scaled L^2 norms of all derivatives of r local interpolating polynomials. An essentially zero weight is assigned to those lower order polynomials whose underlining substencils contain high gradients and/or shocks, aiming at an essentially non-oscillatory solution close to discontinuities. At smooth parts of the solution, the formal order of accuracy is achieved through the mimicking of the central upwind scheme of maximum order, when all smoothness indicators are about the same size. Several techniques have been devised to design the nonlinear weights such as the original weights given in WENO-*JS* [1], the mapped weights given in WENO-*M* [3] and the optimal order weights given in WENO-*Z* [7, 8]. It has been shown that the new set of nonlinear weights of WENO-*Z* provided less dissipation than WENO-*JS* and yielded comparable resolution of smooth solution and captured sharp gradients as WENO-*M* [9–11]. In this study, we will employ the WENO-*Z* for our numerical experiments.

Following [7, 8], the hyperbolicity of the Euler equations admits a complete set of right and left eigenvectors for the Jacobian of the system. The approximated eigenvalues and eigenvectors are obtained via the Roe averaged Jacobian. The first order global Lax-Friedrichs flux is used as the low order building block for the high order reconstruction step of the WENO scheme. After projecting the positive and negative fluxes on the characteristic fields via the left eigenvectors, the high order WENO reconstruction step is applied (first/second) to obtain the high order approximation at the cell boundaries using the surrounding cell-centered values, which are then projected back into the physical space via the right eigenvectors and added together to form a high order numerical flux at the cell-interfaces. The conservative difference of the reconstructed high order fluxes then determines the divergence of the inviscid