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Error Estimates and Superconvergence of Mixed Finite Element Methods for Optimal Control Problems with Low Regularity

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Abstract. In this paper, we investigate the error estimates and superconvergence property of mixed finite element methods for elliptic optimal control problems. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. We derive L^2 and L^∞-error estimates for the control variable. Moreover, using a recovery operator, we also derive some superconvergence results for the control variable. Finally, a numerical example is given to demonstrate the theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: Elliptic equations, optimal control problems, superconvergence, error estimates, mixed finite element methods.

1 Introduction

The finite element approximation of optimal control problems has been extensively studied in the literature. It is impossible to even give a very brief review here. For the studies about convergence and superconvergence of finite element approximations for optimal control problems, see, e.g., [5, 11, 13, 15, 17, 21–25, 28, 30–32]. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, e.g., [9, 19, 20, 29].

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Compared with standard finite element methods, the mixed finite methods have many numerical advantages. When the objective functional contains gradient of the state variable, we will firstly choose the mixed finite element methods. We have done some works on a priori error estimates and superconvergence properties of mixed finite elements for optimal control problems, see, e.g., [3,4,6,8,26]. In [4], we used the postprocessing projection operator, which was defined by Meyer and Rösch (see [21]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, we derived error estimates and superconvergence of mixed methods for convex optimal control problems in [8]. However, in [8], the regularity assumption for the state and the co-state variables is a little strong.

The goal of this paper is to derive the error estimates and superconvergence of mixed finite element approximation for an elliptic control problem. Firstly, by use of the duality argument, we derive the superconvergence property between average L^2 projection and the approximation of the control variable, the convergence order is $h^{3/2}$ as that obtained in [8]. Then the error estimates of order h in the L^2-norm and in the L^∞-norm for the control variable are derived. Moreover, two global superconvergence results with the order $h^{3/2}$ for the control variable can be obtained by using a recovery operator. We can see that the regularity assumption for the state and the co-state variables is only $y,z \in H^2(\Omega)\cap W^{1,\infty}(\Omega)$. Finally, we present a numerical experiment to demonstrate the practical side of the theoretical results.

We consider the following linear optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with pointwise constraint:

$$\min_{u\in U_{ad}} \Big\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\Big\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\nabla y)+a_0 y = u, \qquad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p}+a_0 y=u, \quad \boldsymbol{p}=-A(x)\nabla y, \quad x\in\Omega \tag{1.3}$$

and the boundary condition

$$y=0, \qquad x\in\partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad}=\big\{u\in L^\infty(\Omega):\ u\geq 0, \text{ a.e. in }\Omega\big\}. \tag{1.5}$$

Moreover, we assume that $0\leq a_0\in W^{1,\infty}(\Omega)$, $y_d\in H^1(\Omega)$ and $\boldsymbol{p}_d\in (H^1(\Omega))^2$. ν is a fixed positive number. The coefficient $A(x)=(a_{ij}(x))$ is a symmetric matrix function