Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 6, pp. 848-863

DOI: 10.4208/aamm.12-12S13
December 2012

Numerical Method for the Time Fractional Fokker-Planck Equation

Xue-Nian Cao[1,*], Jiang-Li Fu[1] and Hu Huang[1]

[1] *School of Mathematics and Computational Science, Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Hunan 411105, China*

Revised 26 August 2012; Accepted (in revised version) 3 September 2012

Available online 9 November 2012

Abstract. In this paper, a new numerical algorithm for solving the time fractional Fokker-Planck equation is proposed. The analysis of local truncation error and the stability of this method are investigated. Theoretical analysis and numerical experiments show that the proposed method has higher order of accuracy for solving the time fractional Fokker-Planck equation.

AMS subject classifications: 34A08, 26A33, 65L12, 65L20

Key words: Fractional Fokker-Planck equation, Riemann-Liouville fractional derivative, truncation error, stability.

---

1 Introduction

The Fokker-Planck equation (FPE) is a widely used equation in statistical physics and describes the time evolution of a test particle under the influence of an external force field. The solution of a FPE is the probability of the particle at a certain position at a given time. It has been observed (cf. [3]) that in presence of an highly non-homogeneous medium the anomalous diffusion is not adequately described by the conventional FPE and models based on fractional derivational were proposed. Fractional Fokker-Planck equation (FFPE) can arise in which the temporal derivative and/or spatial derivative operators are fractional, see, e.g., [1, 2, 9–12, 14–16, 25–27]), but in this paper, we only consider fractional derivative operators with respect to time.

Consider the following time fractional Fokker-Planck equation (FFPE)

$$\frac{\partial}{\partial t}P(x,t) = {}_0 D_t^{1-\alpha} \left[\frac{\partial}{\partial x} \frac{U'(x)}{\eta_\alpha} + \kappa_\alpha \frac{\partial^2}{\partial x^2} \right] P(x,t), \tag{1.1}$$

*Corresponding author.
Email: cxn@xtu.edu.cn (X.-N. Cao), fujiangli11@126.com (J.-L. Fu), huanghu1227@163.com (H. Huang)

where $P(x,t)$ denotes the probability density, $U(x)$ indicates the potential of over-damped Brownian motion, κ_α denotes the anomalous diffusion coefficient, η_α represents the generalized friction coefficient, ${}_0D_t^{1-\alpha}P(x,t)$ stands for the Riemann-Liouville fractional derivative of order $1-\alpha$, which is defined by

$${}_0D_t^{1-\alpha}P(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{P(x,\tau)}{(t-\tau)^{1-\alpha}}d\tau.$$

Some works have been done on developing numerical methods for the time FFPE (1.1) in the literature. Deng (cf. [6]) combined the Predictor-Corrector approach with the method of line, and presented a numerical algorithm for solving FFPE with the numerical error $\mathcal{O}(h^{\min(1+2\alpha,2)}) + \mathcal{O}(\tau^2)$ (where and later h is time stepsize, τ is spatial stepsize), and got the corresponding stability condition. Chen et al. (cf. [5]) considered the Grünwald-Letnikov expansion and the L_1-approximation for fractional time derivative, while the first-order spatial derivative is approximated by the backward Euler implicit scheme, or the central or backward difference implicit scheme, then three numerical schemes were given. The local truncation errors $\mathcal{O}(h+\tau)$, $\mathcal{O}(h^{2-\alpha}+\tau^2)$ and $\mathcal{O}(h^{2-\alpha}+\tau)$ were obtained respectively. The objective of this paper is to achieve the efficient numerical methods with higher accuracy.

The outline of this paper is as follows. In Section 2, the numerical method for FFPE is given. Then, in Sections 3 and 4, the analysis of local truncation error and stability of this numerical method are obtained. Section 5 is used to present numerical results, comparing the fixed stepsize implementation of several methods on FFPE problem. The advantages of the new numerical method is readily apparent in computational accuracy and efficiency.

2 Numerical method

In this section, we introduce new numerical scheme for solving the FFPE

$$\frac{\partial}{\partial t}P(x,t) = {}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{U'(x)}{\eta_\alpha} + \kappa_\alpha\frac{\partial^2}{\partial x^2}\right]P(x,t) \tag{2.1}$$

with the initial condition and the boundary conditions

$$P(x,0) = \psi(x), \tag{2.2}$$

$$P(c,t) = \varphi_1(t), \quad P(d,t) = \varphi_2(t), \tag{2.3}$$

where $x \in [c,d], t \in [0,T], 0 < \alpha < 1$, $\psi(x)$, $\varphi_1(t)$ and $\varphi_2(t)$ are given function.

Eq. (2.1) can be written as

$$\frac{\partial}{\partial t}P(x,t) = {}_0D_t^{1-\alpha}\left[\frac{U''(x)}{\eta_\alpha}P(x,t) + \frac{U'(x)}{\eta_\alpha}\frac{\partial}{\partial x}P(x,t) + \kappa_\alpha\frac{\partial^2}{\partial x^2}P(x,t)\right]. \tag{2.4}$$