Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 6, pp. 821-832

DOI: 10.4208/aamm.12-12S11
December 2012

Simulations of Two-Step Maruyama Methods for Nonlinear Stochastic Delay Differential Equations

Wanrong Cao[1,2,*] and Zhongqiang Zhang[2]

[1] *Department of Mathematics, Southeast University, Nanjing 210096, China.*
[2] *Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.*

Revised 31 May 2012; Accepted (in revised version) 20 June 2012

Available online 9 November 2012

Abstract. In this paper, we investigate the numerical performance of a family of P-stable two-step Maruyama schemes in mean-square sense for stochastic differential equations with time delay proposed in [8, 10] for a certain class of nonlinear stochastic delay differential equations with multiplicative white noises. We also test the convergence of one of the schemes for a time-delayed Burgers' equation with an additive white noise. Numerical results show that this family of two-step Maruyama methods exhibit similar stability for nonlinear equations as that for linear equations.

AMS subject classifications: 65C20, 65C30, 65Q20

Key words: P-stability in mean-square sense, two-step Maruyama methods, nonlinear stochastic delay differential system, Burgers' equation.

1 Introduction

We consider numerical schemes for stochastic delay differential equations (SDDEs), which have been increasingly used to model the effects of noise and time delay on various types of complex systems, such as delayed visual feedback systems [5], control problems [14, 24], the dynamics of noisy bi-stable systems with delay [26], etc. SDDEs are also used in modeling diseases [4, 6] and in models of stock markets [15].

Some one-step numerical schemes for SDDEs and their convergence and stability properties have been established recently [3, 13, 17, 20, 27]. Here we focus on stochastic multi-step methods for SDDEs, which can be treated as an nontrivial extension of the multi-step methods of stochastic ordinary differential equations (SODEs), i.e., with no time delay. For an early review of multi-step methods for SODEs, see [16, 22]. Some more recent studies on SODEs can be found in [9] (two-step Maruyama methods), [12]

*Corresponding author.
Email: wrcao@seu.edu.cn (W. Cao), zhongqiang_zhang@brown.edu (Z. Zhang)

(stochastic Adams-Bashforth scheme), [7] (Adams-type schemes), and also [11] (even high order multi-step methods).

Numerical methods for SDDES can also be considered as extension from deterministic delay differential equations (DDEs). A review of numerical methods and their stability for DDEs can be found in [2]. The stability of SDDEs is a bit different from the SODEs, as we may require some conditions on the size of delay and time step size of the numerical methods. For example, P-stability introduced by Barwell [1] refers that the numerical solution of the delay differential equation $y'(t) = ay(t) + by(t-\tau)$ goes to zero when time goes to infinity for any step size h and the time delay $\tau = mh$ (m is an integer), provided that $|b| < -\mathrm{Re}(a)$; see also [25], where Tian and Kuang considered the P-stability of linear multistep methods for DDEs. We will also adopt this concept for the stability of numerical methods for a linear scalar SDDE.

Inspired by [8, 10], we investigate the numerical performance in this paper of a family of two-step Maruyama schemes for a class of the following scalar equation

$$\mathrm{d}X(t) = f\big(t, X(t), X(t-\tau)\big)\mathrm{d}t + g\big(t, X(t), X(t-\tau)\big)\mathrm{d}W(t), \quad t \in J, \tag{1.1a}$$
$$X(t) = \xi(t), \quad t \in [-\tau, 0], \tag{1.1b}$$

where τ is a positive fixed delay, $J = [0, T]$, $W(t)$ is a one-dimensional standard Wiener process and the functions $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In [8], multi-step methods are proposed for m-dimensional systems of Itô SDDEs with d driving Wiener processes and multiple delay, and their properties are studied concerning consistency, numerical stability and convergence. In [10], a series of conditions of parameters of the two-step Maruyama method for SDDEs are given. Under these conditions the family of two-step Maruyama schemes are proved to be P-stable in mean-square sense for a linearized equation of (1.1) as follows:

$$\mathrm{d}X(t) = [aX(t) + bX(t-\tau)]\mathrm{d}t + [cX(t) + dX(t-\tau)]\mathrm{d}W(t), \quad t \geq 0, \tag{1.2a}$$
$$X(t) = \xi(t), \quad t \in [-\tau, 0], \tag{1.2b}$$

where $a, b, c, d \in \mathbb{R}$, τ is a positive fixed delay, $W(t)$ is a one-dimensional standard Wiener process and $\xi(t)$ is a $\mathrm{C}([-\tau, 0]; \mathbb{R})$-valued initial segment. Here we aim to test the aforementioned stability and convergence of two-step Maruyama methods for some scalar or system of nonlinear stochastic differential equations with time delay.

The paper is outlined as follows. In Section 2, we provide some necessary notations and preliminaries on SDDEs, including some properties of analytical solutions to Eq. (1.2). We also introduce in this section the two-step Maruyama methods and their convergence properties and derive a family of P-stable two-step Maruyama methods in mean square sense under certain conditions. Section 3 illustrates the P-stability of these two-step Maruyama methods with numerical examples for a nonlinear delay equation with multiplicative white noises and a stochastic delay differential system. Before we conclude, we compute a Burgers' equation with time delay and additive white noise by some of the proposed schemes and show the mean-square convergence of the scheme.

2 Two-step Maruyama schemes for SDDEs

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$, which satisfies the usual conditions (increasing and right-continuous; each $\{\mathcal{F}_t\}$, $t \geq 0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$).

Let $W(t), t \geq 0$ in Eq. (1.2) be $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. Assume $\xi(t), t \in [-\tau, 0]$ to be $\mathcal{F}_0$-measurable and right continuous, and $E\|\xi\|^2 < \infty$. Here $\|\xi\|$ is defined by $\|\xi\| = \sup_{-\tau\leq t\leq 0} |\xi(t)|$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}$. Throughout the paper, Eqs. (1.2) and (1.1) are interpreted in the Itô sense. Under these assumptions, Eq. (1.2) has a unique strong solution $X(t) : [-\tau, +\infty) \to \mathbb{R}$, which satisfies Eq. (1.2) and $X(t)$ is a measurable, sample-continuous and $\mathcal{F}_t$-adapted process; see [18, 23]. We get the following lemma from [19].

Lemma 2.1. *If there is a positive constant δ such that $\|\xi\| < \delta$ and the condition*

$$a < -|b| - (|c| + |d|)^2 \tag{2.1}$$

holds, then the solution of Eq. (1.2) *is exponentially mean-square stable, i.e., there exist the constants $\lambda > 0$ and $C > 0$, such that*

$$E|X(t, \xi)|^2 \leq CE\|\xi\|^2 e^{-\lambda t}, \quad t \geq 0. \tag{2.2}$$

The inequality (2.2) implies $\lim_{t\to\infty} E|X(t)|^2 = 0$ as we assume $E\|\xi\|^2 < \infty$. Here we extend the P-stability to the numerical method for SDDEs.

Definition 2.2. *A numerical method is said to be P-stable in mean-square sense (or exponentially stable in mean-square), if for all coefficients satisfy the condition* (2.1)*, the numerical solution X_n of Eq.* (1.2) *at the mesh $t_n = nh$, $n \geq 0$ satisfies*

$$\lim_{n\to\infty} E(X_n)^2 = 0$$

for every stepsize h under the constraint $h = \tau/m$, where m is an integer.

The two-step Maruyama methods in [8] for Eq. (1.1) read

$$\begin{aligned}\sum_{j=-1}^{1} \alpha_j X_{i-j} =& h \sum_{j=-1}^{1} \beta_j f(t_{i-j}, X_{i-j}, X_{i-m-j}) \\ &+ \sum_{j=0}^{1} \gamma_j g(t_{i-j}, X_{i-j}, X_{i-m-j})\Delta W_{i-j}, \quad i = 1, 2, \cdots, N,\end{aligned} \tag{2.3}$$

where α_j, β_j, γ_j $(j \in \{-1, 0, 1\})$ are parameters; $h > 0$ is the step size in time which satisfies $\tau = mh$ for a positive integer m, and $t_n = nh$, $N = T/h$. The increments $\Delta W_i := W(t_{i+1}) - W(t_i)$, are independent $\mathcal{N}(0, h)$-distributed Gaussian random variables. Suppose that X_i is $\mathcal{F}_{t_i}$-measurable at the mesh-point t_i. Then X_i is an approximation to $X(t_i)$, where for $i \leq 0$, X_i are given by the initial condition.

Lemma 2.3 (Consistency and convergence, see [8]). *Assume that*

- the coefficients f and g of the SDDE (1.1) are Lipschitz continuous and have first-order continuous partial derivatives with respect to the first variable and second-order continuous partial derivatives with respect to the second and third variables;
- these partial derivatives satisfy the linear growth condition;
- the characteristic polynomial of (2.3), $\rho(\lambda) = \alpha_{-1}\lambda^2 + \alpha_0\lambda + \alpha_1$ has its roots lying on or within the unit circle and the roots are simple if they are on the unit circle,
- and the consistency conditions

$$\sum_{j=-1}^{1} \alpha_j = 0, \quad 2\alpha_{-1} + \alpha_0 = \sum_{j=-1}^{1} \beta_j, \quad \alpha_{-1} = \gamma_0, \quad \alpha_{-1} + \alpha_0 = \gamma_1. \tag{2.4}$$

Then the global error of the scheme (2.3) *for* (1.1) *satisfies*

$$\max_{i=2,\cdots,N} \big(E|X(t_i) - X_i|^2\big)^{\frac{1}{2}} = \mathcal{O}(h^{\frac{1}{2}}).$$

The two-step Maruyama methods (2.3) for the linear equation (1.2) are

$$\begin{aligned}\sum_{j=-1}^{1} \alpha_j X_{i-j} =& h \sum_{j=-1}^{1} \beta_j [aX_{i-j} + bX_{i-m-j}] \\ &+ \sum_{j=0}^{1} \gamma_j [cX_{i-j} + dX_{i-m-j}]\Delta W_{i-j}, \quad i = 1, 2, \cdots,\end{aligned} \tag{2.5}$$

for $i \le 0$, we have $X_i = \xi(t_i)$. By choosing the parameters that satisfy the consistency condition (2.4) in (2.5) and

$$\alpha_{-1} = 1, \quad -1 \le \alpha_0 < 0, \quad \beta_0 = \beta_1 = 0, \tag{2.6}$$

we get

$$\alpha_1 = -1 - \alpha_0, \quad \beta_{-1} = 2 + \alpha_0, \quad \gamma_0 = 1, \quad \gamma_1 = 1 + \alpha_0 \tag{2.7}$$

and thus we obtain a family of two-step Maruyama schemes [10]

$$\begin{aligned}X_{i+1} + \alpha_0 X_i + (-1 - \alpha_0) X_{i-1} =& h(2 + \alpha_0)(aX_{i+1} + bX_{i-m+1}) + (cX_i + dX_{i-m})\Delta W_i \\ &+ (1 + \alpha_0)(cX_{i-1} + dX_{i-m-1})\Delta W_{i-1}\end{aligned} \tag{2.8}$$

from (2.5). We again suppose that X_1 is $\mathcal{F}_{t_1}$-measurable at the mesh-point t_1. The schemes (2.8) have the following mean-square asymptotic stability.

Theorem 2.4 (see [10]). *If the condition* (2.1) *holds and the parameters of the two-step Maruyama method* (2.5) *satisfy the conditions* (2.4) *and* (2.6), *then the method is P-stable in mean-square sense.*

3 Numerical results for nonlinear equations

We list in Table 1 five two-step Maruyama schemes with different parameters under test in this section. Note that the schemes in bold (TS2 and TS4) satisfy the required conditions in Theorem 2.4 and hence are P-stable in mean-square sense; while the other two-step Maruyama schemes TS1, TS3 (Milne-Simpson scheme) and TS5 (Adams-Bashforth 2 scheme) satisfy consistency conditions (2.4), but not the condition (2.6) and they are not P-stable in mean-square sense as we show later on.

Table 1: Parameters for different two-step Maruyama schemes [10].

scheme	α_{-1}	α_0	α_1	β_{-1}	β_0	β_1	γ_0	γ_1
two-step method 1 (TS1)	1	$-1/2$	$-1/2$	$1/4$	$5/4$	0	1	$1/2$
two-step method 2 (TS2)	**1**	**-1/2**	**-1/2**	**3/2**	**0**	**0**	**1**	**1/2**
Milne-Simpson (TS3)	1	0	-1	$1/3$	$4/3$	$1/3$	1	1
two-step method 4 (TS4)	**1**	**-2/3**	**-1/3**	**4/3**	**0**	**0**	**1**	**1/3**
Adams-Bashforth 2 (TS5)	1	-1	0	0	$3/2$	$-1/2$	1	0

In all our numerical examples, the second-order moment

$$E(X_n^2) = \frac{1}{2000}\sum_{i=1}^{2000} |X_n(\omega_i)|^2$$

are the sampled average over 2000 trajectories implemented in Matlab. For better mean-square stability, we compute X_1 up to time t_1 using implicit Milstein method, which is mean-square stable for every $h = \tau/m$ and of order one ($\mathcal{O}(h)$) in the mean-square sense.

Example 3.1. We consider the nonlinear model with two multiplicative white noises

$$\begin{aligned} \mathrm{d}X(t) &= [-5X(t) + 2\sin(X(t-\tau))]\mathrm{d}t + \frac{1}{2}X(t-\tau)\mathrm{d}W_1(t) \\ &\quad - \frac{3}{2}X(t-\tau)\cos(X(t-\tau))\mathrm{d}W_2(t), \qquad t \geq 0, \qquad (3.1a) \\ X(t) &= t + \tau, \qquad t \in [-\tau, 0]. \qquad (3.1b) \end{aligned}$$

By Corollary 2.4, Chapter 6 in [21] (see also Appendix), Eq. (3.1) is exponentially stable in mean-square. Here we test this stability of the two-step Maruyama schemes in Table 1 for Eq. (3.1).

From Fig. 1(a), we observe that TS1 is not mean-square stable for the large step size $h = 1/2$ but is stable for a smaller step size, $h = 1/6$; while in Fig. 1(b), we observe that TS2 is mean-square stable for both step sizes, $h = 1/2$ and $h = 1/6$ and is damping more faster than TS1 for $h = 1/6$. We observe similar effects from TS3 and TS4 in Fig. 2. From Fig. 2, we notice that TS3 is only stable up to time $t = 3$ when $h = 1/4$ is relatively small.

Fig. 3 show that the mean-square stability of the AB2 scheme (TS5) is much better than the implicit two-step schemes TS1 and Milne-Simpson scheme (TS3) even up

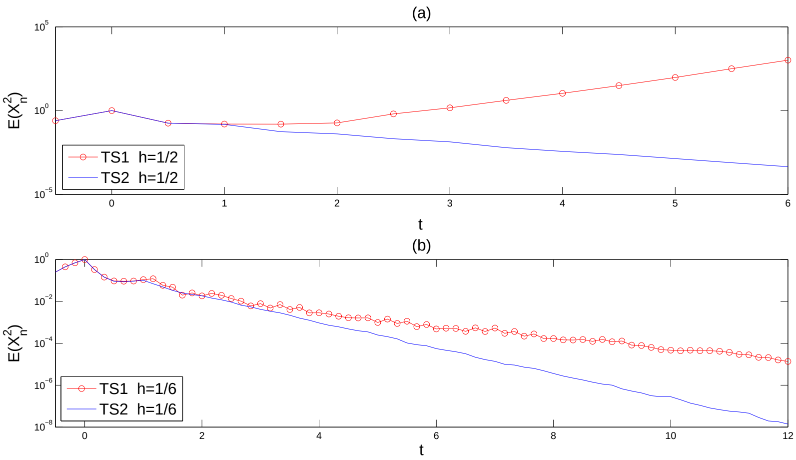

Figure 1: Simulations with TS1 and TS2. (a) $h = 1/2$; (b) $h = 1/6$.

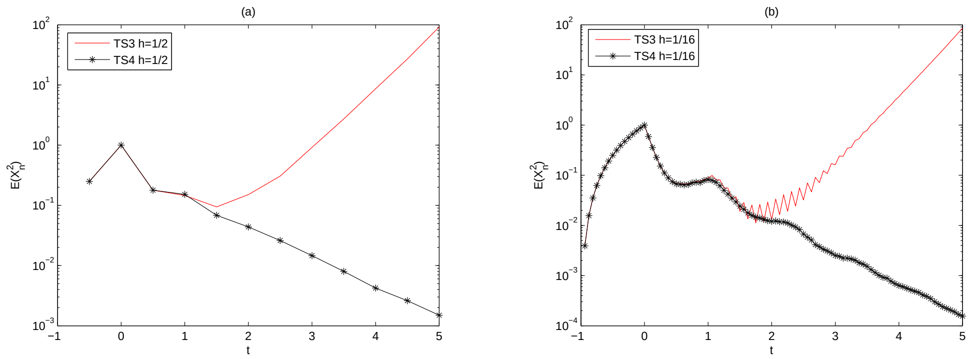

Figure 2: Simulations with TS3 and TS4. (a) $h = 1/2$; (b) $h = 1/16$.

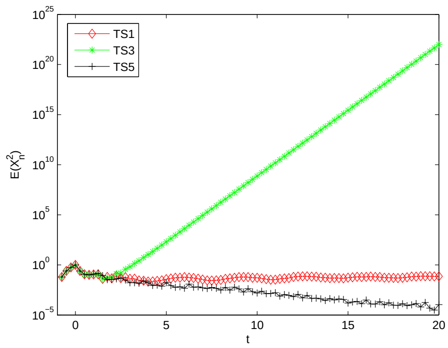

Figure 3: Simulations with TS1, TS3 and TS5. $h = 1/4$.

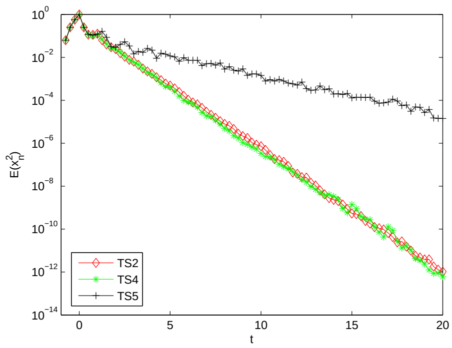

Figure 4: Simulations with TS2, TS4 and TS5. $h = 1/4$.

to very large time. Note that this is contrary to what we observed for the scheme (2.8) for a linear equation, Example 1 in [10]. Compared to the schemes TS2 and TS4, numerical solutions by TS5 converge to zero more slowly than those by TS2 and TS4 when $h = 1/4$, see Fig. 4.

Example 3.2. We test the proposed two-step Maruyama methods for the following nonlinear stochastic delay differential system:

$$\begin{aligned}\begin{pmatrix} \mathrm{d}X_1(t) \\ \mathrm{d}X_2(t) \end{pmatrix} &= \left[A \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} + B \begin{pmatrix} \sin(X_1(t-\tau)) \\ \cos(X_2(t-\tau)) \end{pmatrix} \right] \mathrm{d}t \\ &\quad + C \begin{pmatrix} X_1(t-\tau) \\ X_2(t-\tau) \end{pmatrix} \begin{pmatrix} \mathrm{d}W_1(t) \\ \mathrm{d}W_2(t) \end{pmatrix}, \end{aligned} \tag{3.2}$$

where

$$A = \begin{pmatrix} -28 & 0 \\ 0 & -30 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1/2 \\ 1/4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3/2 \\ 5/2 & -1/2 \end{pmatrix}.$$

The initial condition is given by $X_1(t) = t + \tau$ and $X_2(t) = e^t$ when $t \in [-\tau, 0]$. The solution of system (3.2) is exponentially mean-square stable by Corollary 2.4 [21] (see also Appendix).

From Fig. 5, we observe that TS1 blows up for both $h = 1/4$ and $1/8$ ((a) and (c)) and the P-stable scheme TS2 is mean-square stable, see Figs. 5(b) and (d). It is shown from Fig. 6 that the Milne-Simpson scheme TS3 is not mean-square stable even for small step size $h = 1/64$.

In Fig. 7, we test the P-stable (only proved for linear equations as in Theorem 2.4) schemes TS2 and TS4. With very large step size $h = 1/2$ and $h = 1$, the schemes TS2 and TS4 are maintaining their P-stability in mean-square sense for nonlinear stochastic delay differential system (3.2).

We compare solutions obtained by the schemes TS3 and TS5 in Fig. 8. Fig. 8(a) shows that neither of them are mean-square stable for $h = 1/4$, and the solution from the explicit scheme TS5 blows up faster than that from implicit scheme TS3. On the other hand, the explicit scheme TS5 performs better than the scheme TS3 when $h = 1/16$ as we can see from Fig. 8(b) that TS5 is mean-square stable for $h = 1/16$ but

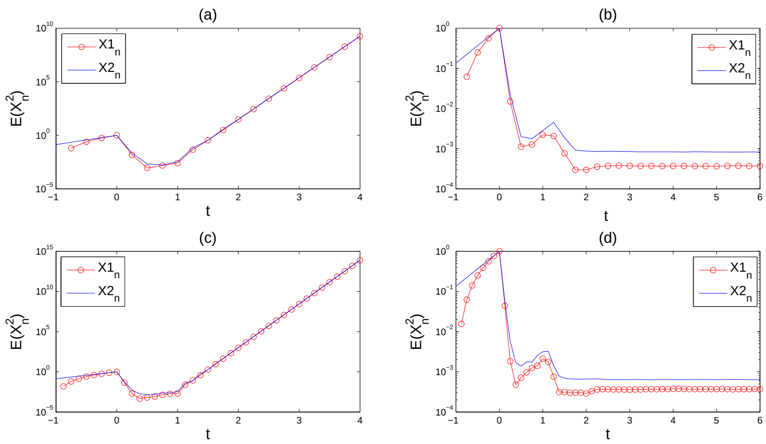

Figure 5: Simulations with TS1 and TS2. (a) TS1, $h = 1/4$; (b) TS2, $h = 1/4$; (c) TS1, $h = 1/8$; (d) TS2, $h = 1/8$.

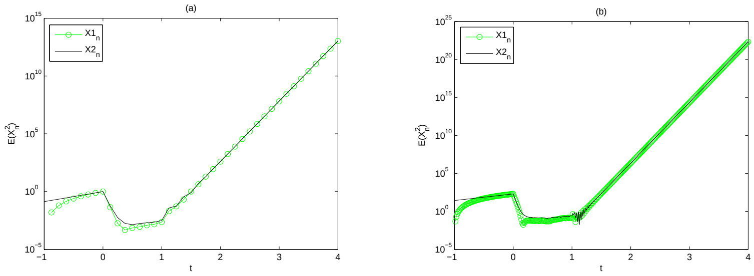

Figure 6: Simulations with TS3. (a) $h = 1/8$; (b) $h = 1/64$.

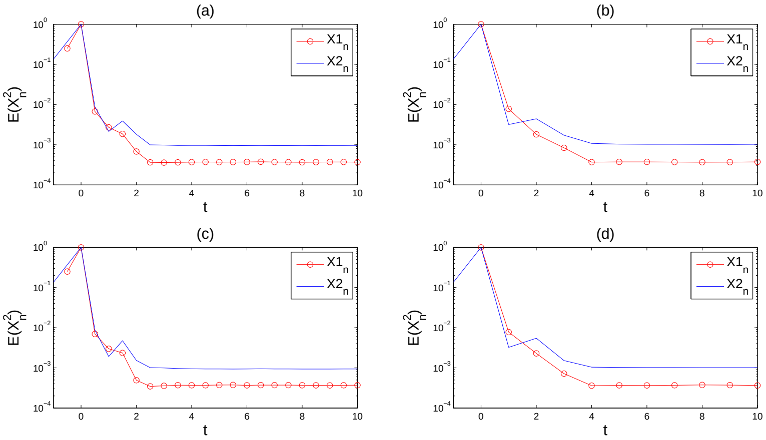

Figure 7: Simulations with TS2 and TS4. (a) TS2, $h = 1/2$; (b) TS2, $h = 1$; (c) TS4, $h = 1/2$; (d) TS4, $h = 1$.

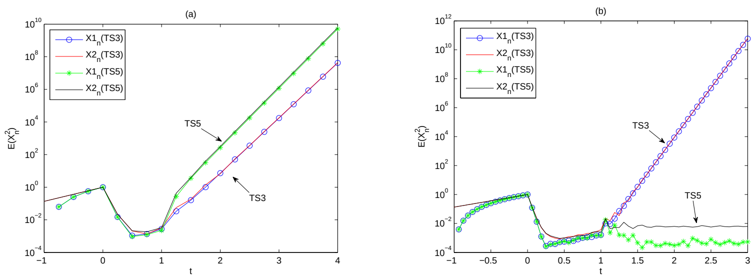

Figure 8: Simulations with TS3 and TS5: (a) $h = 1/4$ and (b) $h = 1/16$.

TS3 is not. Fig. 8 indicates that the explicit scheme TS5 requires less restricted time step size h for mean-square stability than the implicit scheme TS3 does. This effect is exactly what we observe in Example 3.1.

Example 3.3. Consider a time-delayed Burgers' equation with an additive noise [28]

$$
\begin{aligned}
& u_t(x,t) + u(x,t-\tau)u_x(x,t) - u_{xx}(x,t) = \sigma \dot{W}(t), && (x,t) \in (-\pi,\pi) \times (0,T], && \text{(3.3a)} \\
& u(-\pi,t) = u(\pi,t), && t \in [0,T], && \text{(3.3b)} \\
& u(x,t) = \sin(x+t), && (x,t) \in (-\pi,\pi) \times [-\tau,0]. && \text{(3.3c)}
\end{aligned}
$$

Table 2: Convergence test of the scheme (3.4) for Eq. (3.3).

M	h	$e(h)$	$\mathcal{O}\left(\log_2 \frac{e(h)}{e(2h)}\right)$
64	1/64	0.0076	–
64	1/128	0.0039	0.9625
64	1/256	0.0020	0.9635
64	1/512	0.0010	1.0000
64	1/1024	0.0005	0.9979

The parameters in the Eq. (3.3) are $\tau = 1$, $\sigma = 0.01$. We adopt the Fourier collocation method with M points in physical space and the two-step Maruyama scheme (2.8) in time. That is, we solve the problem

$$\begin{aligned}\vec{u}_{k+1} + \alpha_0\vec{u}_k - (1+\alpha_0)\vec{u}_{k-1} =& h(2+\alpha_0)[D^2\vec{u}_{k+1} - \vec{u}_{k-m+1} \circ (D\vec{u}_{k+1})] \\ &+ \sigma(\Delta W_k + (1+\alpha_0)\Delta W_{k-1}),\end{aligned} \tag{3.4}$$

where D is the Fourier spectral differential matrix, "$\circ$" denotes the Hadamard product of matrix and $\vec{u}_k = (u_1^k, \cdots, u_M^k)^T \approx (u(x_1, t_k), \cdots, u(x_M, t_k))^T$, $t_k = kh$.

In (3.4), we take $\alpha_0 = -0.8$ and measure the error of the scheme (3.4) in the following sense:

$$e(h) = \max_k \max_i \Big| \frac{1}{1000} \sum_{l=1}^{1000} u_E(x_i, t_k, \omega_l) - \frac{1}{1000} \sum_{l=1}^{1000} u_i^k(\tilde{\omega}_l) \Big|.$$

The "exact" solution $u_E(x_i, t_k)$ is obtained by the Monte Carlo method with 10^3 realizations and the Fourier collocation method with $M = 64$ points and the Euler-Maruyama method with $h = 10^{-5}$.

We obtain first-order convergence in weak sense of the two-step Maruyama scheme (3.4) as shown in Table 2. Fig. 9 shows the error curves with different time step sizes at $t = 0.5$ and $t = 1$ and Fig. 10 shows the numerical solution by (3.4) with $h = 1/64$ and the "exact" solution at $t = 0.5$ and $t = 1$.

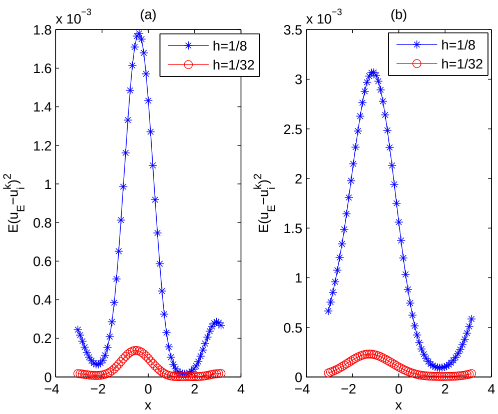

Figure 9: Variances of numerical solutions by (3.4) with different step sizes. (a) $t = 0.5$; (b) $t = 1$.

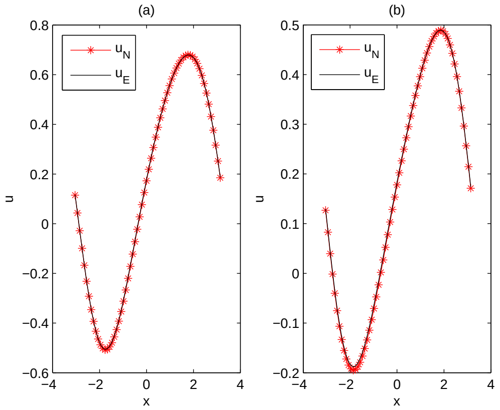

Figure 10: Numerical solution (u_N) with $h = 1/64$ and "Exact" solution (u_E). (a) $t = 0.5$; (b) $t = 1$.

4 Conclusions

We have tested numerically a family of P-stable two-step Maruyama schemes in mean-square sense for a class of nonlinear SDDEs. We tested three cases of nonlinear SDDEs: scalar equation with multiple noises, a system of equation with multiple noises, and a time-delayed Burgers' equation with an additive noise. Numerical simulations show that this family of schemes exhibits mean-square stability for nonlinear stochastic delay differential system. For the time-delayed Burger's equation, we also tested the convergence of our scheme, which is of order one in the weak sense. Numerical results suggest further stability and convergence study of numerical methods for exponentially mean-square stable nonlinear stochastic delay differential system.

As we adopt Monte Carlo method to compute the numerical solution, we always have so-called statistical error, which is proportional to one over the number of sample path. A possible solution to resolve this issue is to adopt stochastic collocation methods or quasi-Monte Carlo methods [29] if numerical solution is required at moderate time. Our future work will also be in this direction.

Appendix

The following lemma assures that Eqs. (3.1) and (3.2) are exponentially stable in mean-square.

$$\mathrm{d}X(t) = f\big(t, X(t), X(t-\tau)\big)\mathrm{d}t + \sum_{k=1}^{\infty} g_k\big(t, X(t), X(t-\tau)\big)\mathrm{d}W_k(t),\ t \geq t_0, \tag{A.1a}$$

$$X(t_0 + s) = \xi(s), \qquad s \in [-\tau, 0], \tag{A.1b}$$

where τ is a positive fixed delay, $W_k(t)$, $k \geq 1$ is a sequence of independent Brownian motions and $f : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $g_k : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. Assume the equation has a unique solution that is denoted by $X(t, t_0, \xi)$.

Lemma A.1 (Corollary 2.4, see [21]). *Let c_1, c_2, c_3, c_4 be positive constants. Assume for all $x, y \in \mathbb{R}^d$ and $t \geq 0$*

$$
\begin{aligned}
&1)\ \ 2x^T f(t,x,0) \leq -c_1|x|^2, \\
&2)\ \ |f(t,x,y) - f(t,x,0)| \leq c_2|y|, \\
&3)\ \ \sum_{k=1}^{\infty} |g_k(t, X(t), X(t-\tau))|^2 \leq c_3|x|^2 + c_4|y|^2,
\end{aligned}
$$

if $c_1 > 2c_2 + c_3 + c_4$, then Eq. (A.1) *is exponentially stable in mean square. That is, there exist constants λ, $C > 0$, such that*

$$E|X(t)|^2 \leq CE\|\xi\|^2 e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

Acknowledgments

This work was supported by the NSF of China (No. 10901036) and AIRFORCE MURI. The authors thank the referees for their helpful suggestions for improving the paper. The first author also would like to thank Professor George Em Karniadakis for his hospitality when she was visiting Division of Applied Mathematics at Brown University.

References

[1] V. K. BARWELL, *Special stability problems for functional equations*, BIT, 15 (1975), pp. 130–135.

[2] A. BELLEN AND M. ZENNARO, Numerical Methods for Delay Differential Equations, Oxford, Clarendon Press, 2003.

[3] C. T. H. BAKER AND E. BUCKWAR, *Exponential stability in p-th mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations*, J. Comput. Appl. Math., 184 (2005), pp. 404–427.

[4] E. BERETTA, V. B. KOLMANOVSKII AND L SHAIKHET, *Stability of epidemic model with time delays influenced by stochastic perturbations*, Math. Comput. Simul., 45 (1998), pp. 269–277.

[5] A. BEUTER AND K. VASILAKOS, *Effects of noise on a delayed visual feedback system*, J. Theor. Bio., 165 (1993), pp. 389–407.

[6] A. BEUTER AND J. BELAIR, *Feedback and delays in neurological diseases: a modeling study using dynamical systems*, Bull Math. Bio., 55 (1993), pp. 525–541.

[7] L. BRUGNANO, K. BURRAGE AND P. M. BURRAGE, *Adams-type methods for the numerical solution of stochastic ordinary differential equations*, BIT, 40 (2000), pp. 451–470.

[8] E. BUCKWAR AND R. WINKLER, *Multi-step Maruyama methods for stochastic delay differential equations*, Stoch. Anal. Appl., 25 (2007), pp. 933–959.

[9] E. BUCKWAR AND R. WINKLER, *Multi-step methods for SDEs and their application to problems with small noise*, SIAM J. Numer. Anal., 44 (2006), pp. 779–803.

[10] W. CAO AND Z. ZHANG, *On mean-square stability of two-step Maruyama methods for stochastic delay differential equations*, J. Comput. Appl. Math., Submitted.

[11] G. DENK AND S. SCHÄFFLER, *Adams methods for the efficient solution of stochastic differential equations with additive noise*, Computing, 59 (1996), pp. 153–161.

[12] B. D. EWALD AND R. TÈMAM, *Numerical analysis of stochastic schemes in geophysics*, SIAM J. Numer. Anal., 42 (2005), pp. 2257–2276.

[13] Z. FAN, *Waveform relaxation method for stochastic differential equations with constant delay*, Appl. Numer. Math., 61 (2011), pp. 229–240.

[14] M. GRIGORIU, *Control of time delay linear systems with Gaussian white noise*, Prob. Eng. Mech., 12 (1997), pp. 89–96.

[15] D. G. HOBSON AND L. C. G. ROGERS, *Complete models with stochastic balatility*, Math. Fina., 8 (1998), pp. 27–48.

[16] P. E. KLOEDEN AND E. PLATEN, Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1992.

[17] R. LI, *Convergence and stability of numerical solutions to SDDEs with Markovian switching*, Appl. Math. Comput., 175 (2006), pp. 1080–1091.

[18] X. MAO, Stochastic Differential Equations and Applications, Horwood, 1997.

[19] X. MAO, *Razumikhin-Type theorems on exponential stability of stochastic functional differential equations*, Stochastic Proc. Appl., 65 (1996), pp. 233–250.

[20] X. MAO, *Numerical solutions of stochastic differential delay equations under the generalized Khasminskii-type conditions*, Appl. Math. Comput., 217 (2011), pp. 5512–5524.

[21] X. MAO, Exponential Stability of Stochastic Differential Equations, Marcel Dekker, INC., New York, 1994.

[22] G. N. MILSTEIN, Numerical Integration of Stochastic Differential Equations, Kluwer Academic Publishers Group, Dordrecht, 1995.

[23] S. E. A. MOHAMMED, Stochastic Functional Differential Equations, Research Notes in Mathematics, Pitman, London, 99 (1984).

[24] M. D. PAOLA AND A. PIRROTTA, *Time delay induced effects on control of linear systems under random excitation*, Prob. Eng. Mech., 16 (2001), pp. 43–51.

[25] H. J. TIAN AND J. X. KUANG, *The numerical stability of linear multistep methods for delay differential equations with many delays*, SIAM J. Numer. Anal., 33 (1996), pp. 883–889.

[26] L. S. TSIMRING AND A. PIKOVSKY, *Noise-induced dynamics in bistable systems with delay*, Phys. Rev. Lett., 87 (2001), pp. 250602-1–25062-4.

[27] S. ZHOU AND F. WU, *Convergence of numerical solutions to neutral stochastic delay differential equations with Markovian switching*, J. Comput. Appl. Math., 229 (2009), pp. 85–96.

[28] W. LIU, *Asymptotic behavior of solutions of time-delayed Burgers' equation*, Dis. Cont. Dyn. Syst. Series B, 2 (2002), pp. 47–56.

[29] Z. ZHANG, B. ROZOVSKII AND G. E. KARNIADAKIS, *Stochastic collocation methods for stochastic differential equations driven by white noise*, Submitted to SISC.