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Bessel Sequences and Its F-Scalability

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Abstract. Frame theory, which contains wavelet analysis and Gabor analysis, has become a powerful tool for many applications of mathematics, engineering and quantum mechanics. The study of extension principles of Bessel sequences to frames is important in frame theory. This paper studies transformations on Bessel sequences to generate frames and Riesz bases in terms of operators and scalability. Some characterizations of operators that mapping Bessel sequences to frames and Riesz bases are given. We introduce the definitions of F-scalable and P-scalable Bessel sequences. F-scalability and P-scalability of Bessel sequences are discussed in this paper, then characterizations of scalings of F-scalable or P-scalable Bessel sequences are established. Finally, a perturbation result on F-scalable Bessel sequences is derived.

AMS subject classifications: 42C15, 46B15

Key words: Frames, Bessel sequences, proper Bessel sequences, F-scalability.

1 Introduction

The concept of frames was first introduced by Duffin and Schaeffer [13]. Then for decades of development both in theory and applications, frame theory are now playing important roles in many applications in mathematics, science and engineering. Frame theory contains wavelet analysis, Gabor analysis and abstract frame theory. Wavelet theory and Gabor analysis are widely used in mathematics, engineering, signal processing and quantum mechanics, see [11,12,16–18]. Finite frame theory is also useful in finite-dimensional quantum mechanics [10,22]. For introductions of fundamental knowledge of frame theory in Hilbert spaces we refer the reader to [4,6], and [8].

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In quantum mechanics, Hilbert spaces (complete inner-product spaces) play a central role in view of the interpretation associated with wave functions [19–21]. Orthonormal basis is a key tool in representations of the state spaces [19]. In the view point of functional analysis, an orthonormal basis gives unique representation of any given object, and therefore would not be able to offer freedoms in representations [11]. Frames serve as a generalized orthonormal basis and offer freedoms on representing the given objects. However, in infinite dimensional Hilbert spaces, it is not easy to verify that a given sequence is a frame or not. To prove that a given sequence is a Bessel sequence or not would be easier [8]. Therefore, the investigation of extension principles of Bessel sequences to frames would be an important topic. In [7], Casazza and Leonhard showed that any Bessel sequence in a finite dimensional space can be extended to a tight frame. Li and Sun extended this result to the infinite dimensional case in [15]. In [9], Christensen et al. showed that in any separable Hilbert space, any pair of Bessel sequences can be extended to a pair of dual frames. Recently, finite extensions of Bessel sequences to frames in infinite dimensional Hilbert spaces was studied by Bakić and Berić [1].

For all the extension principles mentioned above, it is necessary to add new elements, finitely or infinitely, into a given Bessel sequences to generate frames. These extension principles produce better frames, but also lead to additional computational complexity. When one hopes to save computational costs, adding new elements to generate better frames would not be a preferable choice. In this paper we will discuss extension principles of Bessel sequences and frames in terms of operators and scalability in infinite dimensional spaces. Applying operator transformations on Bessel sequences to produce frames can preserve the number of elements in the frames. We find an equivalent condition for operators mapping Bessel sequences to frames in Proposition 3.5. And we prove that, under certain circumstances, operators that mapping Bessel sequences to frames or Riesz bases are unbounded.

Generating tight frames from a given sequence of vectors is also a basic problem in frame theory. The simplest method to generate better frames is by re-scaling the elements in the given complete sequences. Kutyniok et al. introduced scalable frames in [14], a special type of frames which can be modified to generate a Parseval frame by rescaling its frame vectors. Later on, finite scalable frames was investigated in [3] by Cahill and Chen.

We will investigate Bessel sequences which can be scaled to produce frames or Parseval frames. A complete Bessel sequence that can be scaled to generate a frame is called F-scalable in this paper, while it is called P-scalable if it can be scaled to be a Parseval frame. We give some characterizations of F-scalable and P-scalable Bessel sequences, and derive a perturbation result on F-scalable Bessel sequences.

Our paper is organized as follows. In Section 2, we review the needed facts from frame theory and summarize the basic notations in this paper. Section 3 is devoted to the discussions of operators mapping Bessel sequences to frames and Riesz bases. In Section 4, we introduce the definitions of F-scalable and P-scalable Bessel sequences. We investigate the F-scalability and P-scalability of Bessel sequences, then we give charac-