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Optimal Bicubic Finite Volume Methods on Quadrilateral Meshes

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Abstract. In this paper, an optimal bicubic finite volume method is established and analyzed for elliptic equations on quadrilateral meshes. Base on the so-called element-wise stiffness matrix analysis technique, we proceed the stability analysis. It is proved that the new scheme has optimal $\mathcal{O}(h^3)$ convergence rate in H^1 norm. Additionally, we apply this analysis technique to bilinear finite volume method. Finally, numerical examples are provided to confirm the theoretical analysis of bicubic finite volume method.

AMS subject classifications: 65N30, 65N15

Key words: Finite volume method, bicubic element, quadrilateral meshes.

1 Introduction

Finite volume methods (FVMs) [1–14, 16], also known as generalized difference methods [17–24], box schemes [25–27], covolume methods [28], have been one class of the most commonly used numerical methods for solving partial differential equations in practice. Generally speaking, FVMs are flexible to handle complicated domain geometries and boundary conditions. More important, FVMs can preserve the local conservation of certain physical quantities (e.g., mass, momentum, and energy), so they have enjoyed great popularity in many fields, such as computational fluid dynamics, computational aerodynamics, petroleum engineering and so on.

The FVMs for elliptic equations on triangulation have been much studied in the literature [3, 5, 8, 13, 17, 18, 25, 26, 29, 35]. On the other hand, the FVMs on quadrilateral meshes are also commonly used in the physical applications. Many early results have been published on the accuracy of FVM with isoparametric bilinear finite element on

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quadrilateral meshes. The authors of [21] and [27] studied H^1 error estimate and obtained $\|u-u_h\|_1 \leq Ch|u|_2$ independently. In 2007, Yang [10] constructed a second-order finite volume method on quadrilateral meshes. In comparison to Yang [10], Lv [12] established an optimal biquadratic finite volume method on quadrilateral meshes which has optimal $\mathcal{O}(h^2)$ convergence rate in H^1 norm and $\mathcal{O}(h^3)$ convergence rate in L^2 norm. In this paper, we constructed and analyzed an optimal bicubic finite volume method, which is a natural next step in setting up a theoretical basis.

It was known that the stability analysis is a difficult task in error estimates for higher order FVMs. The traditional analysis method is the so-called element stiffness matrix analysis, in which all eigenvalues of an element stiffness matrix often need to be calculated. In order to keep the ellipticity of the bilinear form of FVMs, the minimum eigenvalue of an element stiffness matrix needs to be positive, which requires the mesh to satisfy certain geometric requirements (cf. e.g., [10, 12, 14, 19–21, 35]). Zhang and Zou [31] provide a unified analysis for FVMs of any order on rectangular meshes, in which the stability analysis technique surpasses the so-called elementwise stiffness matrix technique. By a special mapping, the proof of stability avoids calculating eigenvalues of an element stiffness matrix and the global inf-sup condition is established for FVMs of any order.

The remainder of the present paper is organized as follows. In Section 2, we introduce some necessary notations and assumptions, and construct the bicubic finite volume method. Section 3 is devoted to H^1 error estimate. In Section 4, we apply the so-called elementwise stiffness matrix analysis technique to the stability analysis of bilinear finite volume element method. In the last section, numerical examples are presented to confirm the theoretical analysis of bicubic finite volume method.

2 Bicubic finite volume method

In this section we propose the bicubic finite volume method on quadrilateral meshes to solve the following elliptic boundary value problem

$$-\nabla\cdot(A(x,y)\nabla u)=f \qquad \text{in } \Omega, \tag{2.1a}$$

$$u=0 \qquad \text{on } \partial\Omega, \tag{2.1b}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded convex polygon with boundary $\partial\Omega$, and $f\in L^2(\Omega)$. The coefficient $A(x,y)=(a_{ij}(x,y))_{i,j=1}^2$, whose entries $a_{ij}\in C^1(\Omega)$, is assumed to be symmetric, bounded and uniformly positive definite in $\overline{\Omega}$; i.e., there exist two positive constants C_1 and C_2 such that

$$C_1\boldsymbol{\zeta}^T\boldsymbol{\zeta}\leq\boldsymbol{\zeta}^T A(x,y)\boldsymbol{\zeta}\leq C_2\boldsymbol{\zeta}^T\boldsymbol{\zeta}, \quad \forall\boldsymbol{\zeta}\in\mathbb{R}^2, \quad \forall(x,y)\in\overline{\Omega}. \tag{2.2}$$

Divide $\overline{\Omega}$ into a sum of finite number of strictly convex quadrilaterals such that different quadrilaterals have no common interior point, that a vertex of any quadrilateral does not lie on the interior of a side of any other quadrilateral, and that any vertex of