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Nonlinear Stability and B-convergence of Additive Runge-Kutta Methods for Nonlinear Stiff Problems

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Abstract. In this paper, we are devoted to nonlinear stability and B-convergence of additive Runge-Kutta (ARK) methods for nonlinear stiff problems with multiple stiffness. The concept of $(\theta,\bar{p},\bar{q})$-algebraic stability of ARK methods for a class of stiff problems $K_{\sigma,\tau}$ is introduced, and it is proven that this stability implies some contractive properties of the ARK methods. Some results on optimal B-convergence of ARK methods for $K_{\sigma,0}$ are given. These new results extend the existing ones of RK methods and ARK methods in the references. Numerical examples test the correctness of our theoretical analysis.

AMS subject classifications: 65L08, 65L20

Key words: Stiff problem, additive Runge-Kutta method, implicit-explicit method, B-convergence, algebraic stability.

1 Introduction

Consider the initial value problems of stiff ordinary differential equations

$$\begin{cases} y'(t)=f(t,y)=f^{[1]}(t,y(t))+\cdots+f^{[N]}(t,y(t)), & t\in[0,T], \\ y(0)=y_0, \end{cases} \tag{1.1}$$

where $y(t),y_0\in R^m$, f and $f^{[i]}:[0,T]\times R^m\to R^m$ $(i=1,2,\cdots,N)$ are sufficiently smooth vector functions with multiple stiffness. Assume that the problems (1.1) satisfy

$$\begin{aligned} &2\langle f^{[i]}(t,y)-f^{[i]}(t,\tilde{y}),y-\tilde{y}\rangle \\ &\le\sigma_i\|y-\tilde{y}\|^2+\tau_i\|f^{[i]}(t,y)-f^{[i]}(t,\tilde{y})\|^2, \quad i=1,2,\cdots,N, \quad \forall y,\tilde{y}\in R^m, \end{aligned} \tag{1.2}$$

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where σ_i, τ_i $(i=1,2,\cdots,N)$ are real numbers, the norm $\|\cdot\|$ is induced by the standard inner product $\langle\cdot,\cdot\rangle$ on R^m. Let $\sigma=[\sigma_1,\sigma_2,\cdots,\sigma_N]$, $\tau=[\tau_1,\tau_2,\cdots,\tau_N]$. The class of all problems (1.1) satisfying the condition (1.2) is called the class $K_{\sigma,\tau}$. We assume that the true solution $y(t)$ of (1.1) is unique and sufficiently smooth.

In this paper, the symbol $G\geq 0$ $(G>0)$ means that the matrix G is non-negative definite (positive definite), the symbol $x\geq\widetilde{x}$ $(x>\widetilde{x})$ means that the vectors $x=[x_1,x_2,\cdots,x_k]^T$, $\widetilde{x}=[\widetilde{x}_1,\widetilde{x}_2,\cdots,\widetilde{x}_k]^T\in R^k$ satisfy $x_i\geq\widetilde{x}_i$ $(x_i>\widetilde{x}_i)$, $i=1,2,\cdots,k$. $\|G\|$ denotes the norm of the matrix G, which is subject to the vector norm $\|\cdot\|$; $\mu(G)$ denotes the logarithmic matrix norm of G.

For the class $K_{\sigma,\tau}$ being non-empty, it is easy to prove that $\sigma_j\tau_j\leq 1$ when $\tau_j\leq 0$ for $j\in\{1,2,\cdots,N\}$. In fact, if $K_{\sigma,\tau}$ is not empty, then there exists an initial value problem belonging to $K_{\sigma,\tau}$ with the vector function $f^{[i]}(t,y)$ satisfying the condition (1.2), $i=1,2,\cdots,N$. Obviously, $\sigma_j\tau_j\leq 0$ when $\tau_j=0$ or $\tau_j<0$, $\sigma_j\geq 0$ for $j\in\{1,2,\cdots,N\}$. When $\tau_j<0$, $\sigma_j<0$, we have

$$\begin{aligned}
&\sigma_j\|y-\widetilde{y}\|^2+\tau_j\|f^{[j]}(t,y)-f^{[j]}(t,\widetilde{y})\|^2\\
\geq&2\langle f^{[j]}(t,y)-f^{[j]}(t,\widetilde{y}),y-\widetilde{y}\rangle\\
\geq&-2\|y-\widetilde{y}\|\cdot\|f^{[j]}(t,y)-f^{[j]}(t,\widetilde{y})\|\\
\geq&\frac{1}{\tau_j}\|y-\widetilde{y}\|^2+\tau_j\|f^{[j]}(t,y)-f^{[j]}(t,\widetilde{y})\|^2,
\end{aligned}$$

for $\forall t\geq 0$, $\forall y,\widetilde{y}\in R^m$, $y\neq\widetilde{y}$. Thus $\sigma_j\geq 1/\tau_j$, i.e., $\sigma_j\tau_j\leq 1$. This fact for the class $K_{\sigma,\tau}$ with $N=1$ was shown in [25]. Therefore, in this paper, we further assume that the class $K_{\sigma,\tau}$ satisfies the conditions $\sigma_i\tau_i\leq 1$, $i=1,2,\cdots,N$.

In [25], some properties of the class $K_{\sigma,\tau}$ with $N=1$ are given. Now, we extend them to the case $N>1$.

Lemma 1.1. *If $\sigma<0$, then*

$$K_{\sigma,\tau}\subset K_{\sigma-\varepsilon,\tau+\widetilde{M}(\varepsilon,\sigma,\tau)}\quad \textit{for}\ \ \forall\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N)^T\geq 0,$$

where

$$\widetilde{M}(\varepsilon,\sigma,\tau)=(\varepsilon_1\overline{M}(\sigma_1,\tau_1),\varepsilon_2\overline{M}(\sigma_2,\tau_2),\cdots,\varepsilon_N\overline{M}(\sigma_N,\tau_N))^T,\quad \overline{M}(\sigma_i,\tau_i)=\Big(\frac{1+\sqrt{1-\sigma_i\tau_i}}{\sigma_i}\Big)^2.$$

Proof. For any problems belonging to the class $K_{\sigma,\tau}$, Eqs. (1.1)-(1.2) yield

$$\sigma_i\|y-\widetilde{y}\|^2+2\|y-\widetilde{y}\|\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\|+\tau_i\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\|^2\geq 0,$$

and

$$|\sigma_i\|y-\widetilde{y}\|+\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\||\leq\sqrt{1-\sigma_i\tau_i}\|f^{[i]}(t,y)-f^{[i]}(t,\widetilde{y})\|,$$