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Thermoelectric Viscoelastic Fluid with Fractional Integral and Derivative Heat Transfer

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Abstract. A new mathematical model of magnetohydrodynamic (MHD) theory has been constructed in the context of a new consideration of heat conduction with a time-fractional derivative of order $0<\alpha\leq 1$ and a time-fractional integral of order $0<\gamma\leq 2$. This model is applied to one-dimensional problems for a thermoelectric viscoelastic fluid flow in the absence or presence of heat sources. Laplace transforms and state-space techniques [1] will be used to obtain the general solution for any set of boundary conditions. According to the numerical results and its graphs, conclusion about the new theory has been constructed. Some comparisons have been shown in figures to estimate the effects of the fractional order parameters on all the studied fields.

AMS subject classifications: 65M10, 78A48

Key words: Viscoelastic fluid, modified Fourier's and Ohm's laws, modified Riemann-Liouville fractional integral, new fractional Taylor's series, state space approach, fractional calculus.

1 Introduction

Although the tools of fractional calculus have been available and applicable to various fields of study, the investigation of the theory of fractional differential equations has only been started quite recently [2]. The differential equations involving Riemann-Liouville differential operators of fractional order $0<\alpha<1$, appear to be important in modelling several physical phenomena [3] and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations.

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Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic, and it is one reason why fractional calculus has become more and more popular [4].

In the last decade, considerable interest in fractional calculus has been stimulated by the applications in different areas of physics and engineering. The constitutive equations with fractional derivatives have been proved to be a valuable tool to handle viscoelastic properties. In general, these equations are derived from well-known ordinary models via substituting time derivatives of stress and strain by derivatives of fractional order. A very good fit of experimental data is achieved when the constitutive equation with fractional derivative is used [5]. Some applications of fractional calculus to various problems of mechanics of viscoelastic fluid are reviewed in the literature [6–10]. In most of these investigations the effect of the thermal state in the fluid flow not taking into account.

The modification of the heat conduction equation from diffusive to a wave type may be affected either by a microscopic consideration of the phenomenon of heat transport or in a phenomenological way by modifying the classical Fourier law of heat conduction. The first is due to Cattaneo [11], who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier law. Puri and Kythe [12] investigated the effects of using the (Maxwell-Cattaneo) model in Stock's second problem for a viscous fluid and they note that, the non-dimensional thermal relaxation time to defined as $\tau_o = CP_r$, where C and P_r are the Cattaneo and Prandtl number, respectively, is of order 10^{-2}.

Recently, a considerable research effort has been expended to study anomalous diffusion, which is characterized by the time fractional diffusion-wave equation by Kimmich [13]:

$$\rho c = \varsigma I^{\gamma} \nabla^2 c, \quad 0 < \gamma \leq 2, \tag{1.1}$$

where the notion I^{γ} is the Riemann–Liouville fractional integral is introduced as a natural generalization of the well-known n-fold repeated integral $I^n f(t)$ written in a convolution-type form as [14,15]

$$\left.\begin{aligned} &I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\xi)^{\gamma-1} f(\xi) d\xi, \\ &I^0 f(t) = f(t), \end{aligned}\right\} \quad 0 < \gamma \leq 2, \tag{1.2}$$

where $\Gamma(\gamma)$is the gamma function.

According to Kimmich [13], Eq. (1.1) describes different cases of diffusion where $0 < \gamma < 1$ corresponds to weak diffusion (subdiffusion), $\gamma = 1$ corresponds to normal diffusion, $0 < \gamma < 2$ corresponds to strong diffusion (superdiffusion), and $\gamma = 2$ corresponds