Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **8**, No. 4, pp. 517-535

DOI: 10.4208/aamm.2014.m834
August 2016

Unified a Priori Error Estimate and a Posteriori Error Estimate of CIP-FEM for Elliptic Equations

Jianye Wang* and Rui Ma

LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, China

Received 9 November 2014; Accepted (in revised version) 7 April 2015

Abstract. This paper is devoted to a unified a priori and a posteriori error analysis of CIP-FEM (continuous interior penalty finite element method) for second-order elliptic problems. Compared with the classic a priori error analysis in literature, our technique can easily apply for any type regularity assumption on the exact solution, especially for the case of lower H^{1+s} weak regularity under consideration, where $0 \leq s \leq 1/2$. Because of the penalty term used in the CIP-FEM, Galerkin orthogonality is lost and Céa Lemma for conforming finite element methods can not be applied immediately when $0 \leq s \leq 1/2$. To overcome this difficulty, our main idea is introducing an auxiliary C^1 finite element space in the analysis of the penalty term. The same tool is also utilized in the explicit a posteriori error analysis of CIP-FEM.

AMS subject classifications: 65N30, 65N15, 35J25

Key words: Finite element methods, continuous interior penalty, priori error estimate, posteriori error analysis.

1 Introduction

Finite element methods (FEMs), first suggested in the structural analysis in the fifth decade of last century, have become one of the most important numerical methods in the field of scientific and engineering computing. Penalty methods, initially introduced by Lions [14], were used to impose the solutions of elliptic boundary value problems (BVPs) to satisfy the Dirichlet boundary condition therein. Nitsche and Babuska first applied the penalty technique for solving the elliptic boundary value problem by using the continuous finite element method [5, 15]. After that using penalty technique in the finite element discretization for BVPs to control the discontinuity of the finite element function across the interior triangulation edges developed rapidly. For example, Babuska and Zlámal dealt with the plate bending problem by using C^0 finite element space

*Corresponding author.
Email: jianguoyeluo@163.com (J. Y. Wang), marui1988.happy@163.com (R. Ma)

with interior penalty technique [6], Zienkiewicz also discussed interior penalty FEMs for the fourth order problems for which the trial functions though continuous, are not contained in C^1 [20]. Today penalty methods have become a basic technique in discontinuous Galerkin (DG) methods [4,18].

In 1976, Jim Douglas. Jr. and Todd Dupont first proposed CIP-FEM by adding a penalty term which penalized the numerical flux across the interior edges to the classic C^0 continuous finite element discretization for linear elliptic and parabolic problems in [12]. Controlling the jump of the normal derivative across the interior edge of adjacent elements can produce an apparent stiffness intermediate between C^0 and C^1 finite element spaces when we use Galerkin methods based on C^0 piecewise polynomial spaces. Numerical experiments with that method have clearly demonstrated the value of penalties for solving certain problems which have proved intractable to more conventional methods [12]. Recently CIP-FEMs have been successfully applied for the Helmholtz problem, which has a highly sign-indefinite and rapidly changing solution on the whole domain [19].

Concerning the penalty term of the jumps of flux, the classic error analysis of CIP-FEM for elliptic BVPs requires a priori regularity $H^{\frac{3}{2}+s}$, where $s>0$ is arbitrary. In this case, we could obtain the Galerkin orthogonality and apply Céa Lemma to its error analysis, just playing the trick likewise in the standard finite element error analysis (see, e.g., [12]). However, when the exact solution does not have so much smooth property, the above methods do not work. This paper is dealing with this difficulty by introducing an auxiliary C^1 conforming finite element space, which gives proofs for both a priori error analysis and a posteriori error analysis for CIP-FEM.

The rest of paper is organized as follows. In Section 2 we will give a brief description of CIP-FEM and give some basic lemmas that are fundamental in the following derivation of a priori error analysis and a posteriori error analysis. In Section 3 and Section 4, a priori error estimate and a posteriori error estimate are presented, respectively. Some numerical experiments will be given to demonstrate our theoretical analysis in the last section.

Throughout the paper, C always denotes a positive constant independent of the mesh size and it may not be the same in different places. For convenience, the symbol $\lesssim$ will be used where we can replace $X\leq CY$ by $X\lesssim Y$ for some positive constant C that is independent of mesh size.

2 Preliminaries and notations

Let Ω be a bounded polyhedral domain with boundary $\Gamma=\partial\Omega$ in n dimensional space $\mathbb{R}^n$. For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$, set $|\alpha|=\sum_{i=1}^n\alpha_i$. The derivative operator can be written as

$$\partial^\alpha=\frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1}\cdots\partial {x_n}^{\alpha_n}}.$$