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# Integrable Couplings of the Boiti-Pempinelli-Tu Hierarchy and Their Hamiltonian Structures

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**Abstract.** Integrable couplings of the Boiti-Pempinelli-Tu hierarchy are constructed by a class of non-semisimple block matrix loop algebras. Further, through using the variational identity theory, the Hamiltonian structures of those integrable couplings are obtained. The method can be applied to obtain other integrable hierarchies.

**AMS subject classifications**: 37K05, 35Q53

**Key words**: Integrable coupling, bi-integrable coupling, Hamiltonian structure, block matrix loop algebra.

## 1 Introduction

The research of integrable Hamiltonian systems [1] has undergone a rapid development in recent years. So far as, the most efficient approach is the trace identity [2–4], which was proposed by G. Z. Tu. The crucial step of using the Tu scheme is to design suitable isospectral problems and loop algebras based on a Lie algebra. In general, an integrable hierarchy can be generated by the following steps.

Firstly, one can choose a proper eigenvalue problem and its auxiliary problem

$$\phi_x = U(u,\lambda)\phi, \quad \phi_{t_m} = V^{(m)}(u,\lambda)\phi, \tag{1.1}$$

where $U(u,\lambda)$ is a matrix and $V^{(m)}(u,\lambda)$ are a sequence of matrices, $u=u(u_1,u_2,\cdots,u_n)$ contained in $U(u,\lambda)$ and $V^{(m)}(u,\lambda)$ with $u_i$, $i=1,2,\cdots,n$, being potentials belongs to Schwartz space, $\lambda$ is a spectral parameter and $\lambda_t=0$, and $\phi=(\phi_1,\phi_2)^T$ is a vector eigenfunction.

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Secondly, based on the integrability condition: $\phi_{xt_m} = \phi_{t_m x}$, i.e.,

$$U(u,\lambda)_{t_m} - V^{(m)}(u,\lambda)_x + [U(u,\lambda), V^{(m)}(u,\lambda)] = 0, \tag{1.2}$$

usually a hierarchy of soliton equations is determined as follows:

$$u_{t_m} = K_m(u), \tag{1.3}$$

where $K_m(u)$ is a vector function of $u$ and its $x$-derivatives $u_x, u_{xx}, \cdots$.

Another important problem in the theory of infinite-dimensional integrable systems is to search for a Hamiltonian operator $J$ and a hierarchy of conserved densities $H_m$ so that the Eq. (1.3) can be rewritten into their Hamiltonian forms

$$u_{t_m} = K_m(u) = J\frac{\delta H_m}{\delta u}, \quad m \geq 0, \tag{1.4}$$

which can be furnished by the variational identity theory. Based on the Tu scheme, some experts made a series of achievements [5–14]. In [5, 6], the author further developed Tu scheme and presented new hierarchies of soliton equations. The variational identity over nonsemisimple Lie algebras was developed and applied in [7, 8].

Along with the research of the multi-component trace identities, the theory of integrable coupling and bi-integrable coupling were presented recently [15, 16]. For an integrable evolution equation

$$u_t = K(u) = K(x,t,u,u_x,u_{xx},\cdots), \tag{1.5}$$

its integrable coupling means

$$\overline{u}_t = \overline{K}_1(\overline{u}) = \begin{bmatrix} K(u) \\ S(u,v) \end{bmatrix}, \quad \overline{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \tag{1.6}$$

which is called nonlinear, if $S(u,v)$ is nonlinear with respect to the sub-vector $v$ of dependent variables, and similarly its bi-integrable coupling means

$$\overline{\overline{u}}_t = \overline{\overline{K}}_1(\overline{\overline{u}}) = \begin{bmatrix} K(u) \\ S_1(u,v) \\ S_2(u,v,w) \end{bmatrix}, \quad \overline{\overline{u}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{1.7}$$

In this paper, we aim mainly to explore bi-integrable couplings of the Boiti-Pempinelli-Tu (BPT) hierarchy and its Hamilton structure using the follow zero curvature equation,

$$U_t - V_x + [U,V] = 0 \tag{1.8}$$

and using the block loop matrix approach.

This paper is divided into five sections. In the next section, the BPT hierarchy [2,11,17] is reviewed. Integrable coupling and bi-integrable coupling of the BPT hierarchy are derived, in Section 3 and Section 4 respectively. We conclude this paper in Section 5.