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Semidefinite Optimization Estimating Bounds on Linear Functionals Defined on Solutions of Linear ODEs

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Abstract. In this paper, semidefinite optimization method is proposed to estimate bounds on linear functionals defined on solutions of linear ordinary differential equations (ODEs) with smooth coefficients. The method can get upper and lower bounds by solving two semidefinite programs, not solving ODEs directly. Its convergence theorem is proved. The theorem shows that the upper and lower bounds series of linear functionals discussed can approach their exact values infinitely. Numerical results show that the method is effective for the estimation problems discussed. In addition, in order to reduce calculation amount, Cheybeshev polynomials are applied to replace Taylor polynomials of smooth coefficients in computing process.

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Key words: Semidefinite optimization, bound, linear functional, ordinary differential equation.

1 Introduction

Semidefinite optimization is a very efficient optimization method [1]. It has been successfully used to solve many important problems, including combinatorial optimization [2,3], filter design [4], eigenvalue optimization [5] and mobile communication optimization [6]. Recently, the method was applied to deal with moment problem [7–9], polynomial global optimization [7, 10–17] and bounds on linear functionals defined on solutions of linear differential equations [18] with polynomial coefficients. Instead of finding out the solutions of linear differential equations, D. Bertsimas and C. Caramanis [18] presented semidefinite optimization to estimate bounds on linear functionals defined on solutions of linear differential equations with polynomial coefficients. The numerical results show

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that the method is very efficient for the related problems. The method can easily obtain good upper and lower bounds of the functionals discussed under a small amount computational burden. At this regard, the discretization-based approaches cannot provide guaranteed bounds on functionals defined on solutions of the differential equations.

In this paper, we will propose semidefinite optimization method for estimating bounds on linear functionals defined on solutions of ODEs with general smooth coefficients. The primal problem is first transformed into a series of approximate problems with polynomial coefficients. Then semidefinite optimization method is applied to solve these approximate problems. Convergence theorem of the proposed method is proved. Two numerical examples are given to show the effectiveness of our estimation method. Our approach generalizes the one in [18] in terms of ODEs case, which can estimate bounds on linear functionals defined on solutions of ODEs with general smooth coefficients, including ODEs with polynomial coefficients.

The rest of this paper is organized as follows. In Section 2, we propose semidefinite optimization method for estimating bounds on the linear functionals defined on solutions of ODEs with smooth coefficients. In Section 3, we prove that the approach is convergent for the problems discussed. Numerical experiments are performed in Section 4. We end the paper with some conclusions and discussions in Section 5.

2 Semidefinite optimization

Suppose that L and G are two linear differential operators operating on some distribution space $\mathcal{A}$, in which the space $\mathcal{F}=Span\{1,x,x^2,\cdots,x^i,\cdots\}$ is dense. Their orders are r_1 and r_2 $(r_2\le r_1)$, respectively.

We define L and G as follows:

$$Lu(x)=\sum_{s\le r_1} L_s(x)\frac{d^s u(x)}{dx^s}, \tag{2.1a}$$

$$Gu(x)=\sum_{s\le r_2} G_s(x)\frac{d^s u(x)}{dx^s}, \tag{2.1b}$$

where $L_s(x)$ and $G_s(x)$ are all in $\mathbb{C}^\infty[0,1]$.

Primal Problem: Computing

$$\int_0^1 Gu(x), \tag{2.2}$$

where $u(x)\in\mathcal{A}$ satisfies the following differential equation

$$Lu(x)=f(x), \quad x\in[0,1], \tag{2.3}$$

with some appropriate boundary conditions at $x=0$ and $x=1$. In (2.3), we also suppose $f(x)\in\mathcal{A}$.