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Spectral-Collocation Method for Volterra Delay Integro-Differential Equations with Weakly Singular Kernels

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Abstract. A spectral Jacobi-collocation approximation is proposed for Volterra delay integro-differential equations with weakly singular kernels. In this paper, we consider the special case that the underlying solutions of equations are sufficiently smooth. We provide a rigorous error analysis for the proposed method, which shows that both the errors of approximate solutions and the errors of approximate derivatives decay exponentially in L^∞ norm and weighted L^2 norm. Finally, two numerical examples are presented to demonstrate our error analysis.

AMS subject classifications: 65R20, 65M70, 45D05

Key words: Volterra integro-differential equations, spectral Jacobi-collocation method, pantograph delay, weakly singular kernel.

1 Introduction

In this paper, we consider the following weakly singular Volterra integro-differential equations with pantograph delays

$$\begin{aligned} y'(t) &= a(t)y(t)+b(t)y(qt)+f(t)+\int_0^t (t-s)^{-\mu}K_1(t,s)y(s)ds \\ &\quad +\int_0^{qt}(qt-\tau)^{-\mu}K_2(t,\tau)y(\tau)d\tau, \quad t\in I:=[0,T], \qquad (1.1a) \\ y(0) &= y_0, \qquad (1.1b) \end{aligned}$$

where $0<\mu<1$, $0<q<1$, $y(t)$ is the unknown function, the functions $a,b,f\in C(I)$ and $K_1,K_2\in C(I\times I)$ with $K_1(t,t),K_2(t,qt)\neq 0$ for $t\in I$.

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It is well known that spectral methods have excellent error properties and provide exponential rates of convergence for smooth problems. In the last decades, they have become a class of important methods to numerically solve certain partial differential equations. And there are many existing works on this subject (see e.g., books [3–5, 17, 18] and references therein). Recently, several authors have applied spectral methods to solve Volterra-type integral or integro-differential equations. In [28], spectral and pseudo-spectral Jacobi-Galerkin approaches were introduced to solve the second kind Volterra integral equations. In [20], spectral and pseudo-spectral Jacobi-Petrov-Galerkin approaches were developed to solve the second kind Volterra integro-differential equations. Wan, Chen and Huang [21] used the spectral Galerkin method to solve the nonlinear Volterra integral equations of the second kind. The authors in [1, 9, 10, 19, 22, 23, 26, 31] proposed the spectral Legendre-collocation method for Volterra integral or integro-differential equations with smooth kernels. In [6–8, 12, 24, 25, 29, 30, 32] the spectral Jacobi-collocation method was successfully applied to solve Volterra integral or integro-differential equations with weakly kernels and fractional integro-differential equations. And the spectral accuracy of the approaches is verified both theoretically and numerically in [1, 6–10, 12, 19–26, 28–32].

The numerical treatment of (1.1a)-(1.1b) is not simple, mainly due to the fact that the solutions of (1.1a)-(1.1b) usually have a weak singularity at $t=0$. In this work, we will consider the special case that the exact solutions of (1.1a)-(1.1b) are smooth (see [2]). The main purpose of this paper is to develop a spectral Jacobi-collocation method for Volterra integro-differential equations with weakly singular kernels and pantograph delays based on the works of [23] and [32]. This paper is arranged as follows. In Section 2, we describe the spectral methods for (1.1a)-(1.1b). Some useful lemmas for derivation of the error estimates are given in Section 3. The error estimates in L^∞ norm and weighted L^2 norm of the approximation scheme (2.12a)-(2.12c) are provided in Section 4. Section 5 contains two numerical experiments which are used to verify the theoretical results obtained in Section 4. Finally, in Section 6, we end with the conclusion and the future work.

2 Jacobi-collocation method

Let $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ be a weight function in the usual sense, for $\alpha,\beta > -1$. As defined in [3–5, 17, 18], the set of Jacobi polynomials $\{J_n^{\alpha,\beta}(x)\}_{n=0}^\infty$ forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$-orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1)=\{v:v \text{ is measurable and } \|v\|_{\omega^{\alpha,\beta}}<\infty\},$$

equipped with the norm

$$\|v\|_{\omega^{\alpha,\beta}}=\Big(\int_{-1}^1 |v(x)|^2\omega^{\alpha,\beta}(x)dx\Big)^{\frac{1}{2}},$$