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Two-Level Stabilized Finite Volume Methods for the Stationary Navier-Stokes Equations

Tong Zhang* and Shunwei Xu

School of Mathematics & Information Science, Henan Polytechnic University, Jiaozuo 454003, Henan, China

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Abstract. In this work, two-level stabilized finite volume formulations for the 2D steady Navier-Stokes equations are considered. These methods are based on the local Gauss integration technique and the lowest equal-order finite element pair. Moreover, the two-level stabilized finite volume methods involve solving one small Navier-Stokes problem on a coarse mesh with mesh size H, a large general Stokes problem for the Simple and Oseen two-level stabilized finite volume methods on the fine mesh with mesh size $h=\mathcal{O}(H^2)$ or a large general Stokes equations for the Newton two-level stabilized finite volume method on a fine mesh with mesh size $h=\mathcal{O}(|\log h|^{1/2}H^3)$. These methods we studied provide an approximate solution $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ with the convergence rate of same order as the standard stabilized finite volume method, which involve solving one large nonlinear problem on a fine mesh with mesh size h. Hence, our methods can save a large amount of computational time.

AMS subject classifications: 65N30, 65N08, 76D05

Key words: Stationary Navier-Stokes equations, finite volume method, two-level method, error estimate.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$ and to satisfy a further condition recalled in (**A1**) below. In this work, we consider the steady incompressible Navier-Stokes equations

$$\begin{cases} -\nu\Delta u+(u\cdot\nabla)u+\nabla p=f, & \text{in } \Omega,\\ u=0, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

*Corresponding author.
Email: tzhang@hpu.edu.cn (T. Zhang), xushunwei@hpu.edu.cn (S. W. Xu)

where $u=(u_1(x),u_2(x))^T$ represents the velocity, $p=p(x)$ the pressure, $f=f(x)$ the prescribed body force and $\nu>0$ the viscosity.

The development of efficient mixed finite element methods for the Navier-Stokes equations is an important but challenging problem in incompressible flow simulations. The importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying the so-called inf-sup condition is widely known. Although some stable mixed finite element pairs have been studied over the years [15,26], the P_1-P_1 pair not satisfying the inf-sup condition may also works well. The P_1-P_1 pair is computationally convenient in a parallel processing and multigrid context because this pair holds the identical distribution for both the velocity and pressure. Moreover, the P_1-P_1 pair is of practical importance in scientific computation with the lowest computational cost. Therefore, much attention has been attracted by the P_1-P_1 pair for simulating the incompressible flow, we can refer to [3, 11, 17, 23, 32, 33] and the references therein.

In order to use the P_1-P_1 pair, various stabilized techniques have been proposed and studied. For example, the Brezzi-Pitkaranta method [4], the stream upwind Petrov-Galerkin (SUPG) method [6], the polynomial pressure projection method [11], the Douglas-Wang method [12] and the macro-element method [18]. Most of these stabilized methods necessarily need to introduce the stabilization parameters either explicitly or implicitly. In addition, some of these techniques are conditionally stable or are of suboptimal accuracy. Therefore, the development of mixed finite element methods free from stabilization parameters has become increasingly important.

Recently, a family of stabilized finite element method for Stokes problem has been established in [3] by using a polynomial pressure projection, authors not only presented the stabilized discrete formulation for Stokes equations but also obtained the optimal error estimates. Compared with other stabilized methods which mentioned above, this new stabilized method has following features: parameter-free, avoiding higher-order derivatives or edge-based data structures and unconditionally stable. Based on the ideas of [3, 11], by using the difference of two local Gauss integrations as the component for the pressure, Li et al. developed a kind of stabilized method for linear mixed finite element pair (see [22–25]), and their method can be applied to the existing codes with a little additional effort.

Finite volume method (FVM) as one of important numerical discretization techniques has been widely employed to solve the fluid dynamics problems [14]. It is developed as an attempt to use finite element idea in the finite difference setting. The basic idea is to approximate discrete fluxes of a partial differential equation using the finite element procedure based on volumes or control volumes, so FVM is also called box scheme, general difference method [1, 14]. FVM has many advantages that belong to finite difference or finite element method, such as, it is easy to set up and implement, conserve mass locally and FVM also can treat the complicated geometry and general boundary conditions flexibility. However, the analysis of FVM lags far behind that of finite element and finite difference methods, we can refer to the literature [13, 22, 25, 27, 31] and the reference therein for more recent developments about the finite volume method.