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Convergent Overdetermined-RBF-MLPG for Solving Second Order Elliptic PDEs

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Abstract. This paper deals with the solvability and the convergence of a class of unsymmetric Meshless Local Petrov-Galerkin (MLPG) method with radial basis function (RBF) kernels generated trial spaces. Local weak-form testings are done with step-functions. It is proved that subject to sufficiently many appropriate testings, solvability of the unsymmetric RBF-MLPG resultant systems can be guaranteed. Moreover, an error analysis shows that this numerical approximation converges at the same rate as found in RBF interpolation. Numerical results (in double precision) give good agreement with the provided theory.

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Key words: Local integral equations, meshless methods, radial basis functions, overdetermined systems, solvability, convergence.

1 Introduction

In recent years, there is a rapid growth in research of different variants of meshless methods. Generally speaking, meshless methods for solving partial differential equations (PDEs) can be classified into two groups: one uses the strong-form collocation while another group uses the weak formulation for testing the PDEs. The meshless local Petrov-Galerkin method (MLPG), which was first proposed by Atluri and colleagues in 1998 [1, 2], belongs to the latter group. Since then, the MLPG method has been successfully applied to solve a wide range of problems in engineering and science; see also references [3–6] therein. To see some general properties of the unsymmetric meshless

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kernel based methods see [7]. For a brief introduction to the original MLPG method, let us consider a PDE in the form of:

$$Lu = f \text{ in } \Omega, \quad \text{and} \quad u = g \text{ on } \partial\Omega, \tag{1.1}$$

where $\partial\Omega$ denotes the boundary of the bounded domain Ω in $\mathbb{R}^d$. Both given functions f and $g : \mathbb{R}^d \to \mathbb{R}$ are sufficiently smooth. For any set of N scattered nodal points in the domain and on the boundary represented by $\Xi = \{\xi_k\}_{k=1}^N$. The unknown solution u is approximated via

$$u(x) \approx u_N(x) = \sum_{\boldsymbol{\xi}_i \in \Xi} \lambda_i \Phi_i(x), \tag{1.2}$$

where $\Phi_i(x)$, i=1,2,$\cdots$,N are called shape functions constructed on the set of nodal points Ξ and λ_i is the unknown coefficient at node i to be determined. For certain shape functions, e.g., the moving least-squares basis [8–10], we have weights $\lambda_i \approx u(x_i)$ approximating the solution values. To solve for the N unknowns $\lambda_1,\cdots,\lambda_N$, the "local" weak equations constructed on subdomains surrounding each node are as follows:

$$\int_{\Omega_{s^i}} (Lu) v dx = \int_{\Omega_{s^i}} f v dx \Rightarrow \sum_{j=1}^{N} \Big(\int_{\Omega_{s^i}} L\Phi_j(x) v dx \Big) \lambda_j = \int_{\Omega_{s^i}} f v dx. \tag{1.3}$$

This yields N equations for the N unknown coefficients. In (1.3), Ω_{s^i} denotes a (relatively small) subdomain in Ω surrounding the node x_i and v is a locally supported test function. Employing different test functions v results in different kinds of MLPG methods; see [1, 2]. One possible class of the test functions, that is of our interest in this work, is the step functions. In this paper, we study the solvability and convergence of an MLPG method using radial basis functions (RBF) as shape functions [11–13] and step functions as test functions. Some numerical demonstrations are given to show the exponential convergence (under double-precision computations) of the RBF-MLPG method.

2 Sufficient condition for solvability

In the original MLPG method, the sets of test and trial nodes are identical. Such linkage between these two sets of nodes will be decoupled in the RBF-MLPG method due to the requirement for solvability given in this section. Moreover, we use more test equations (denoted by M) than the number of basis in expansion (denoted by N) to yield overdetermined MLPG systems.

We assume that the differential equation (1.1) has an exact solution u^* lying in some infinite dimensional trial spaces $\mathcal{U}$. To obtain a numerical procedure, we first discretize the trial space $\mathcal{U}$ by some finite dimensional subspaces $\mathcal{U}_N$ generated by a set RBF kernel Φ centered at a set of N scattered nodes (or RBF centers) $\Xi_N := \{\boldsymbol{\xi}_i\}_1^N$. Any numerical approximations are of the form

$$u_N(x) = \sum_{\boldsymbol{\xi}_i \in \Xi} \lambda_i \Phi(x, \boldsymbol{\xi}_i) \in \mathcal{U}_N := \Big\{ v : v(x) = \sum_{\boldsymbol{\xi}_i \in \Xi} \lambda_i \Phi(x, \boldsymbol{\xi}_i), \ \lambda \in \mathbb{R}^N \Big\},$$