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Two-Level Stabilized Finite Volume Methods for the Stationary Navier-Stokes Equations

Tong Zhang* and Shunwei Xu

School of Mathematics & Information Science, Henan Polytechnic University, Jiaozuo 454003, Henan, China

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Abstract. In this work, two-level stabilized finite volume formulations for the 2D steady Navier-Stokes equations are considered. These methods are based on the local Gauss integration technique and the lowest equal-order finite element pair. Moreover, the two-level stabilized finite volume methods involve solving one small Navier-Stokes problem on a coarse mesh with mesh size H, a large general Stokes problem for the Simple and Oseen two-level stabilized finite volume methods on the fine mesh with mesh size $h=\mathcal{O}(H^2)$ or a large general Stokes equations for the Newton two-level stabilized finite volume method on a fine mesh with mesh size $h=\mathcal{O}(|\log h|^{1/2}H^3)$. These methods we studied provide an approximate solution $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ with the convergence rate of same order as the standard stabilized finite volume method, which involve solving one large nonlinear problem on a fine mesh with mesh size h. Hence, our methods can save a large amount of computational time.

AMS subject classifications: 65N30, 65N08, 76D05

Key words: Stationary Navier-Stokes equations, finite volume method, two-level method, error estimate.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$ and to satisfy a further condition recalled in (**A1**) below. In this work, we consider the steady incompressible Navier-Stokes equations

$$\begin{cases} -\nu\Delta u+(u\cdot\nabla)u+\nabla p=f, & \text{in } \Omega, \\ u=0, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

*Corresponding author.
Email: tzhang@hpu.edu.cn (T. Zhang), xushunwei@hpu.edu.cn (S. W. Xu)

where $u=(u_1(x),u_2(x))^T$ represents the velocity, $p=p(x)$ the pressure, $f=f(x)$ the prescribed body force and $\nu>0$ the viscosity.

The development of efficient mixed finite element methods for the Navier-Stokes equations is an important but challenging problem in incompressible flow simulations. The importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying the so-called inf-sup condition is widely known. Although some stable mixed finite element pairs have been studied over the years [15,26], the P_1-P_1 pair not satisfying the inf-sup condition may also works well. The P_1-P_1 pair is computationally convenient in a parallel processing and multigrid context because this pair holds the identical distribution for both the velocity and pressure. Moreover, the P_1-P_1 pair is of practical importance in scientific computation with the lowest computational cost. Therefore, much attention has been attracted by the P_1-P_1 pair for simulating the incompressible flow, we can refer to [3, 11, 17, 23, 32, 33] and the references therein.

In order to use the P_1-P_1 pair, various stabilized techniques have been proposed and studied. For example, the Brezzi-Pitkaranta method [4], the stream upwind Petrov-Galerkin (SUPG) method [6], the polynomial pressure projection method [11], the Douglas-Wang method [12] and the macro-element method [18]. Most of these stabilized methods necessarily need to introduce the stabilization parameters either explicitly or implicitly. In addition, some of these techniques are conditionally stable or are of suboptimal accuracy. Therefore, the development of mixed finite element methods free from stabilization parameters has become increasingly important.

Recently, a family of stabilized finite element method for Stokes problem has been established in [3] by using a polynomial pressure projection, authors not only presented the stabilized discrete formulation for Stokes equations but also obtained the optimal error estimates. Compared with other stabilized methods which mentioned above, this new stabilized method has following features: parameter-free, avoiding higher-order derivatives or edge-based data structures and unconditionally stable. Based on the ideas of [3, 11], by using the difference of two local Gauss integrations as the component for the pressure, Li et al. developed a kind of stabilized method for linear mixed finite element pair (see [22–25]), and their method can be applied to the existing codes with a little additional effort.

Finite volume method (FVM) as one of important numerical discretization techniques has been widely employed to solve the fluid dynamics problems [14]. It is developed as an attempt to use finite element idea in the finite difference setting. The basic idea is to approximate discrete fluxes of a partial differential equation using the finite element procedure based on volumes or control volumes, so FVM is also called box scheme, general difference method [1, 14]. FVM has many advantages that belong to finite difference or finite element method, such as, it is easy to set up and implement, conserve mass locally and FVM also can treat the complicated geometry and general boundary conditions flexibility. However, the analysis of FVM lags far behind that of finite element and finite difference methods, we can refer to the literature [13, 22, 25, 27, 31] and the reference therein for more recent developments about the finite volume method.

On the other hand, two-level method is an efficient numerical scheme for partial differential equations based on two spaces with different mesh sizes. This kind of discretization technique for linear and nonlinear elliptic problems was first introduced by Xu in [28,29]. After then, this scheme has been studied by many researchers, for example, Dawson et al. studied the nonlinear parabolic equations by using the finite element and finite difference methods in [9, 10], respectively. Layton and Lenferink [20] for Navier-Stokes equations, Bi and Ginting [2] have expanded two-level method combined with finite volume method for linear and nonlinear elliptic problems.

Motivated by [18], in this paper we will devote ourselves to the study of the two-level stabilized finite volume methods for the steady Navier-Stokes problem. By introducing a projection between the linear space and constant space, stability and convergence of the finite volume solution are established. Compared with [18], the difference lies in some cases: (i) the formulations are different, it is finite volume method in this work; (ii) the stabilized approach is different from one of [18]; and (iii) the finite element spaces for the pressure are different, it is linear space in our method. The test function is constant in finite volume method, which produces some difficulties to the analysis of stability and convergence, such as, the trilinear term does not satisfy the antisymmetry, the optimal error estimates require $f\in H^1$, or the optimal error estimate in L^2-norm of velocity requires $u\in H^3$. The important novel ingredient of this work is the convergence analysis of the approximate solution of two-level methods. We provide the convergence of $(\widetilde{u}_H^v,\widetilde{p}_H^v)$ of the Navier-Stokes problem. Then the fine mesh approximation $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ is obtained by solving a general large Stokes problem for the Simple and Oseen two-level finite volume methods on a fine mesh with mesh size $h=\mathcal{O}(H^2)$ or a large Stokes problem for the Newton two-level finite volume method on a fine mesh with mesh size $h=\mathcal{O}(|\log h|^{1/2}H^3)$.

For the finite volume solution (u_h^v,p_h^v), which involves solving one large nonlinear problem on a fine mesh with mesh size h, we provide the following error estimate:

$$\|u-u_h^v\|_1+\|p-p_h^v\|_0\le Ch, \tag{1.2}$$

where $C>0$ denotes some generic constant which it may stand for different values at its different occurrences. Furthermore, we prove that the Simple and Oseen two-level finite volume solution $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ is of the following error estimate, respectively:

$$\|u-\widetilde{u}_h^v\|_1+\|p-\widetilde{p}_h^v\|_0\le C(h+H^2). \tag{1.3}$$

Also, we prove that the Newton two-level finite volume solution $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ is of the following error estimate:

$$\|u-\widetilde{u}_h^v\|_1+\|p-\widetilde{p}_h^v\|_0\le C(h+|\log h|^{\frac{1}{2}}H^3). \tag{1.4}$$

Hence, if we choose H such that $h=\mathcal{O}(H^2)$ for Simple and Oseen two-level finite volume solutions and $h=\mathcal{O}(|\log h|^{1/2}H^3)$ for the Newton two-level finite volume solution, then the methods we studied are of the convergence rate of same order as the standard finite volume method. However, our methods are more simple.

2 Function setting for the Navier-Stokes equations

For the mathematical setting of problem (1.1), we set

$$X=H_0^1(\Omega)^2, \qquad Y=L^2(\Omega)^2,$$
$$D(A)=H^2(\Omega)^2\cap X, \qquad M=L_0^2(\Omega)=\Big\{q\in L^2(\Omega): \int_\Omega q dx=0\Big\}.$$

The spaces $L^2(\Omega)^m$ (m=1,2,4) are endowed with the standard L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. The spaces $H_0^1(\Omega)$ and X are equipped with the scalar product $(\nabla u,\nabla v)$ and norm $\|u\|_1^2=(\nabla u,\nabla u)$, $\forall u,v\in H_0^1$ or X.

We introduce the Laplace operator $Au=-\Delta u$, $\forall u\in D(A)$. As mentioned above, we need a further assumption on Ω which provided in [19].

(**A1**) Assume that Ω is smooth so that the unique solution $(v,q)\in X\times M$ of the steady Stokes problem

$$-\Delta v+\nabla q=g, \quad \text{div} v=0 \text{ in } \Omega, \quad v|_{\partial\Omega}=0,$$

for any prescribed $g\in Y$ exists and satisfies

$$\|v\|_2+\|q\|_1\le C\|g\|_0.$$

From Assumption (**A1**), it is well known that (see [5,7])

$$\|v\|_0\le C_1\|v\|_1, \qquad \forall v\in X, \tag{2.1a}$$
$$\|v\|_1\le C_1\|Av\|_0, \quad \|v\|_{L^\infty}\le C_2\|v\|_0^{\frac{1}{2}}\|v\|_2^{\frac{1}{2}}, \quad \forall v\in D(A). \tag{2.1b}$$

We introduce the bilinear forms

$$a(u,v)=\nu(\nabla u,\nabla v), \qquad \forall u,v\in X,$$
$$d(v,p)=(\text{div}v,p), \qquad \forall v\in X, \quad p\in M,$$

and the trilinear form

$$\begin{aligned} b(u,v,w)&=((u\cdot\nabla)v,w)+\frac{1}{2}((\text{div}u)v,w)\\ &=\frac{1}{2}((u\cdot\nabla)v,w)-\frac{1}{2}((u\cdot\nabla)w,v), \quad \forall u,v,w\in X. \end{aligned}$$

It is easy to verify that the trilinear term $b(\cdot,\cdot,\cdot)$ satisfies the following important properties (see [15,26])

$$b(u,v,w)=-b(u,w,v), \quad b(u,v,v)=0, \quad |b(u,v,w)|\le C_3\|u\|_1\|v\|_1\|w\|_1, \tag{2.2}$$

for all $u,v,w \in X$ and

$$|b(u,v,w)|+|b(v,u,w)|+|b(w,u,v)| \leq C_3 \|u\|_1 \|Av\|_0 \|w\|_0, \tag{2.3}$$

for all $u \in X$, $v \in D(A)$, $w \in Y$.

With above notations, the variational formulation of problem (1.1) reads as: Find $(u,p) \in (X,M)$ such that

$$B((u,p);(v,q))+b(u,u,v)=(f,v), \quad \forall (v,q) \in (X,M), \tag{2.4}$$

where $B((u,p);(v,q))=a(u,v)-d(v,p)+d(u,q)$.

Detailed results on existence and uniqueness of the solution to problem (1.1) are provided in [15,26]. In particular, we need to recall the following theorem.

Theorem 2.1. *Under the Assumption* (**A1**)*, if* $\nu > 0$ *and* $f \in Y$ *satisfy*

$$1-\frac{C_1 C_3}{\nu^2}\|f\|_0 > 0, \tag{2.5}$$

then the solution (u,p) *of problem* (2.4) *is unique, and satisfying*

$$\|u\|_1 \leq \frac{C_1}{\nu}\|f\|_0, \quad \|u\|_2+\|p\|_1 \leq C\|f\|_0,$$

where the positive constants C_1 *and* C_3 *are given by* (2.1) *and* (2.2).

3 Stabilized finite element method

This section is devoted to present the stability and convergence of the stabilized finite element solution for the steady Navier-Stokes problem.

Let $\mathcal{T}_h=\{K\}$ be a regular, quasi-uniform partition of the domain Ω into a finite number of triangulations, $h_K=\mathrm{diam}(K)$, $h=\max\{h_K : K \in \mathcal{T}_h\}$, $\mathcal{N}_h$ denotes the set of all nodes $\mathcal{T}_h$. We consider the following mixed finite element spaces

$$X_h=\{v \in C^0(\overline{\Omega})^2 \cap X : v_i|_K \in P_1(K),\ \forall K \in \mathcal{T}_h,\ i=1,2\},$$
$$M_h=\{q \in C^0(\overline{\Omega}) \cap M : q|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\},$$

where $P_1(K)$ is the set of linear polynomials on K.

For the above finite element spaces X_h and M_h, it is well-known that the following approximate estimates hold (see [7,8]).

(**A2**) For $(v,q) \in (D(A), H^1(\Omega) \cap M)$, there exist approximations $I_h v \in X_h$ and $J_h q \in M_h$ such that

$$\|v-I_h v\|_0+h\|v-I_h v\|_1 \leq Ch^2\|Av\|_0, \ \|I_h v\|_1 \leq \|v\|_1, \tag{3.1a}$$
$$\|q-J_h q\|_0 \leq Ch\|q\|_1. \tag{3.1b}$$

Due to the quasi-uniformness of the triangulation $\mathcal{T}_h$, the following properties hold (see [5])

$$\|v_h\|_1 \le C_4 h^{-1}\|v_h\|_0, \quad \|v_h\|_{L^\infty} \le C_5|\log h|^{\frac{1}{2}}\|v_h\|_1, \quad \forall v_h \in X_h. \tag{3.2}$$

For the subsequent analysis, we now introduce a discrete analogue A_h of Laplace operator A through the condition

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h), \quad \forall u_h, v_h \in X_h.$$

Define

$$V_h = \{v_h \in X_h : d(v_h, q_h) = 0, \ \forall q_h \in M_h\}.$$

The restriction of A_h to V_h is invertible. In addition, A_h is self-adjoint and positive definite. This discrete laplace operator is first introduced in [19] to analyze and obtain the optimal estimates for the transient Navier-Stokes equations. By the way, we derive from (2.1) that

$$\|v_h\|_0 \le C_1\|v_h\|_1, \quad \forall v \in X_h; \quad \|v_h\|_1 \le C_1\|A_h v_h\|_0, \quad \forall v_h \in V_h. \tag{3.3}$$

The following estimates about the trilinear form $b(\cdot,\cdot,\cdot)$ can be found in [16].

Lemma 3.1. *The trilinear form $b(\cdot,\cdot,\cdot)$ satisfies the following estimates:*

$$\begin{aligned} &b(u_h, v_h, w_h) = -b(u_h, w_h, v_h), \quad b(u_h, v_h, v_h) = 0, \qquad \forall u_h, v_h, w_h \in X_h, \\ &|b(u_h, v_h, w_h)| \le C|\log h|^{\frac{1}{2}}\|\nabla u_h\|_0\|\nabla v_h\|_0\|w_h\|_0, \qquad \forall u_h, v_h, w_h \in X_h, \end{aligned}$$

and

$$\begin{aligned} &|b(u_h, v_h, w)| + |b(v_h, u_h, w)| + |b(w, v_h, u_h)| \\ \le &\frac{C}{2}\|A_h v_h\|_0^{\frac{1}{2}}\|\nabla v_h\|_0^{\frac{1}{2}}\|u_h\|_0^{\frac{1}{2}}\|\nabla u_h\|_0^{\frac{1}{2}}\|w\|_0 + \frac{C}{2}\|A_h v_h\|_0^{\frac{1}{2}}\|\nabla v_h\|_0^{\frac{1}{2}}\|\nabla u_h\|_0\|w\|_0, \end{aligned}$$

for all $u_h, v_h \in V_h$, $w \in Y$.

Obviously, the lowest equal-order conforming finite element pair does not satisfy the discrete inf-sup condition (see [26])

$$\beta\|q_h\|_0 \le \sup_{0 \ne v_h \in X_h} \frac{d(v_h, q_h)}{\|v\|_1}, \quad \forall q_h \in M_h, \tag{3.4}$$

where the constant $\beta > 0$ is independent of h. In order to overcome the restriction (3.4), we define the L^2-projection operator $\Pi_h : L^2(\Omega) \to W_h$ by (see [3, 24])

$$(p, q_h) = (\Pi_h p, q_h), \quad \forall p \in L^2(\Omega), \quad q_h \in W_h, \tag{3.5}$$

where $W_h \in L^2(\Omega)$ denotes the piecewise constant space associated with the triangulation $\mathcal{T}_h$. The operator Π_h has the following properties (see [23])

$$\|\Pi_h p\|_0 \le C\|p\|_0, \quad \forall p \in L^2(\Omega); \quad \|p-\Pi_h p\|_0 \le Ch\|p\|_1, \quad \forall p \in H^1(\Omega). \tag{3.6}$$

With the help of (3.5), we define a bilinear form $G_h(\cdot,\cdot)$ by

$$G_h(p_h,q_h) = (p_h - \Pi_h p_h, q_h) = (p_h - \Pi_h p_h, q_h - \Pi_h q_h), \quad \forall p_h, q_h \in M_h. \tag{3.7}$$

Remark 3.1. The bilinear form $G_h(\cdot,\cdot)$ in (3.7) is symmetric, semi-positive definite form generated on each local set K.

With the above notation, the finite element variational formulation of (2.4) for the Navier-Stokes equations is recast as: Find $(u_h,p_h) \in X_h \times M_h$, such that

$$B_h((u_h,p_h);(v_h,q_h)) + b(u_h,u_h,v_h) = (f,v_h), \quad \forall (v_h,q_h) \in (X_h,M_h), \tag{3.8}$$

where

$$B_h((u_h,p_h);(v_h,q_h)) = a(u_h,v_h) - d(v_h,p_h) + d(u_h,q_h) + G_h(p_h,q_h).$$

The following theorem establishes the continuity and weak coercivity properties of the generalized bilinear form B_h (see [3,24]).

Theorem 3.1. *Let (X_h,M_h) be defined as above, for all $(u_h,p_h),(v_h,q_h) \in (X_h,M_h)$, there exists a positive constant β_1, independent of h, such that*

$$|B_h((u_h,p_h);(v_h,q_h))| \le C(|u_h|_1 + \|p_h\|_0)(|v_h|_1 + \|q\|_0), \tag{3.9a}$$

$$\beta_1(|u_h|_1 + \|p_h\|_0) \le \sup_{0 \ne (v_h,q_h) \in (X_h,M_h)} \frac{|B_h((u_h,p_h);(v_h,q_h))|}{|v_h|_1 + \|q_h\|_0}. \tag{3.9b}$$

By choosing $(v_h,q_h) = (u_h,p_h) \in X_h \times M_h$ in (3.8) and using Lemma 3.1, we obtain

$$\|\nabla u_h\|_0 \le \frac{C_1}{\nu}\|f\|_0. \tag{3.10}$$

Moreover, we take $v_h = A_h u_h \in V_h$ and $q_h = 0$ in (3.8) and use Lemma 3.1 and Eq. (3.10) to get that

$$\begin{aligned}
\nu\|A_h u_h\|_0 &\le \|f\|_0 + CC_1^{\frac{1}{2}}\|\nabla u_h\|_0^{\frac{3}{2}}\|A_h u_h\|_0^{\frac{1}{2}} \\
&\le \|f\|_0 + \frac{C^2 C_1}{2\nu}\|\nabla u_h\|_0^3 + \frac{\nu}{2}\|A_h u_h\|_0 \\
&\le \frac{\nu}{2}\|A_h u_h\|_0 + \Big(1 + \frac{C^2 C_1^4}{2\nu^4}\|f\|_0^2\Big)\|f\|_0.
\end{aligned} \tag{3.11}$$

The next optimal error estimate holds for the stabilized finite element solution for the steady Navier-Stokes equations (3.8).

Theorem 3.2. (*see* [17]) *Under the Assumptions* (**A1**)*, problem* (3.8) *admits a unique solution and satisfies the following error estimate for sufficiently small $h > 0$*

$$\|u-u_h\|_0 + h(\|u-u_h\|_1 + \|p-p_h\|_0) \le Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0). \tag{3.12}$$

4 Stabilized finite volume method

Based on the partition $\mathcal{T}_h$, we introduce the corresponding dual partition $\mathcal{T}_h^*$. Here, we choose the barycenter Q of a element $K\in\mathcal{T}_h$, and the midpoints M on the edges of K, then connect Q to M by straight line. For an arbitrary vertex $x_i\in K$, let $\widetilde{K}_i$ be the polygonal which is called a control volume. Then, we have

$$\overline{\Omega}=\cup_{x_i\in\mathcal{N}_h}\widetilde{K}_i,$$

the dual mesh $\mathcal{T}_h^*$ is the set of these control volumes.

The dual finite element space is defined as

$$\widetilde{X}_h=\big\{\widetilde{v}\in(L^2(\Omega))^2:\widetilde{v}\in P_0^2(\widetilde{K}_i),\ \forall\widetilde{K}_i\in\mathcal{T}_h^*;\ \ \widetilde{v}|_{\partial\widetilde{K}_i}=0\big\}.$$

It is clearly that the dimensions of X_h and $\widetilde{X}_h$ are the same, and there exists an invertible linear mapping $\Gamma_h:X_h\rightarrow\widetilde{X}_h$ such that

$$\Gamma_h v_h(x)=\sum_{i=1}^{\mathcal{N}_h}v_h(x_i)\phi_i(x),\qquad x\in\Omega,\quad v_h\in X_h,$$

where $\phi_i(x)$ is the basis functions associated with the dual partition $\mathcal{T}_h^*$:

$$\phi_i(x)=\begin{cases}1, & x\in\widetilde{K}_i,\\ 0, & \text{otherwise}\cup x\in\partial\Omega.\end{cases}$$

The above idea of connecting the different spaces through the mapping Γ_h was introduced by Li in [21] for the elliptic problem. Furthermore, the mapping Γ_h has the following properties (see [25,27]).

Lemma 4.1. *Let $K\in\mathcal{T}_h$. If $v_h\in X_h$ and $1\le r\le\infty$, then*

$$\begin{aligned}&\int_K(v_h-\Gamma_h v_h)dx=0,\\ &\|v_h-\Gamma_h v_h\|_{L^r(K)}\le C_6 h_K\|v_h\|_{W^{1,r}(K)},\\ &\|\Gamma_h v_h\|_0\le C_7\|v_h\|_0,\end{aligned}$$

where h_K is the diameter of the element K. In order to simple the analysis of the convergence of two-level schemes, we always assume the constant $C_7\ge1$.

With the help of the Green's formula, the stabilized finite volume method for problem (1.1) reads as: Find $(u_h^v,p_h^v)\in(X_h,M_h)$, $\forall(v_h,q_h)\in(X_h,M_h)$ such that

$$\widetilde{B}_h((u_h^v,p_h^v),(v_h,q_h))+b(u_h^v,u_h^v,\Gamma_h v_h)=(f,\Gamma_h v_h),\tag{4.1}$$

where

$$\widetilde{B}_h((u_h^v,p_h^v),(v_h,q_h)) = A(u_h^v,\Gamma_h v_h) + D(\Gamma_h v_h,p_h^v) + d(u_h^v,q_h) + G(p_h^v,q_h),$$

and

$$\begin{aligned} A(u_h^v,\Gamma_h v_h) &= -\nu\sum_{j=1}^{\mathcal{N}_h} v_h(P_j)\int_{\partial\widetilde{K}_j}\frac{\partial u_h^v}{\partial n}dx, && u_h^v,v_h\in X_h,\\ D(\Gamma_h v_h,p_h^v) &= \sum_{j=1}^{\mathcal{N}_h} v_h(P_j)\int_{\partial\widetilde{K}_j} p_h^v n dx, && p_h\in M_h,\\ (f,\Gamma_h v_h) &= \sum_{j=1}^{\mathcal{N}_h} v_h(P_j)\int_{\widetilde{K}_j} f dx, && v_h\in X_h. \end{aligned}$$

The next lemma establishes the relationship between the finite element and finite volume methods for the Navier-Stokes equations (see [22, 30]).

Lemma 4.2. *It holds that*

$$A(u_h,\Gamma_h v_h) = a(u_h,v_h),\quad \forall u_h,v_h\in X_h.$$

Moreover, the bilinear form $D(\cdot,\cdot)$ satisfies

$$D(q_h,\Gamma_h v_h) = -d(q_h,v_h),\quad \forall (v_h,q_h)\in (X_h,M_h).$$

The following lemma, which has been presented in [22], establishes the continuity and weak coercivity for the general bilinear form $\widetilde{B}_h((u_h,p_h),(v_h,q_h))$.

Lemma 4.3. *It holds that for all $(u_h,p_h)\in(X_h,M_h)$*

$$|\widetilde{B}_h((u_h,p_h),(v_h,q_h))| \le C(\|\nabla u_h\|_0 + \|p_h\|_0)(\|\nabla v_h\|_0 + \|q_h\|_0),\ \ \forall (v_h,q_h)\in(X_h,M_h).$$

Moreover,

$$\beta_2(\|\nabla u_h\|_0 + \|p_h\|_0) \le \sup_{0\neq(v_h,q_h)\in(X_h,M_h)} \frac{|\widetilde{B}_h((u_h,p_h);(v_h,q_h))|}{\|\nabla v_h\|_0 + \|q_h\|_0},$$

where the constant $\beta_2>0$ is independent of h.

By applying the Brouwer fixed point theorem, Li et al. have provided the stability and convergence for the stabilized finite volume solution (u_h^v,p_h^v) for problem (4.1).

Theorem 4.1. *(see* [25]*) For the mesh size $h>0$, if $\nu>0, f\in Y$ satisfy*

$$0 < \frac{4C_1C_5C_6C_7}{\nu^2}|\log h|^{\frac{1}{2}} h\|f\|_0 \le \frac{1}{4} \quad \textit{and} \quad \frac{C_1C_3C_7}{\nu^2}\|f\|_0 \le \frac{1}{8}, \tag{4.2}$$

then the system (4.1) *admits a unique solution* (u_h^v,p_h^v). *Furthermore, it satisfies*

$$\|\nabla u_h^v\|_0 \le \frac{2C_1C_7}{\nu}\|f\|_0,$$
$$\|A_h u_h^v\|_0 \le \frac{2C_7}{\nu}\|f\|_0\Big(1+\frac{2^5C_1^4C_2^2C_7^4}{\nu^4}\|f\|_0^2\Big).$$

Theorem 4.2. (*see* [25]) *Under the condition of* (4.2) *and* $\nu>0$ *and* $f\in Y$ *satisfy*

$$1-\frac{2C_1C_3C_7}{\nu^2}\|f\|_0 \ge C_8>0, \tag{4.3}$$

let $(u,p)\in(X,M)$ *and* $(u_h^v,p_h^v)\in(X_h,M_h)$ *be the solution of* (2.4) *and* (4.1), *respectively. Then it holds*

$$\|\nabla(u-u_h^v)\|_0+\|p-p_h^v\|_0 \le Ch(\|u\|_2+\|p\|_1+\|f\|_0).$$

Furthermore, if $f\in H^1(\Omega)^2$, *then*

$$\|u-u_h^v\|_0 \le Ch^2(\|u\|_2+\|p\|_1+\|f\|_1).$$

5 Two-level stabilized finite volume approximations

From now on, H and $h\ll H$ will be two real positive parameters tending to 0. Also, a coarse mesh triangulation of $\mathcal{T}_H(\Omega)$ of Ω is made as like in Section 3. A fine mesh triangulation $\mathcal{T}_h(\Omega)$ is generated by a mesh refinement process to $\mathcal{T}_H(\Omega)$. The finite element spaces (X_h,M_h) and $(X_H,M_H)\subset(X_h,M_h)$ are based on the triangulations $\mathcal{T}_h(\Omega)$ and $\mathcal{T}_H(\Omega)$, respectively. With the above finite element spaces, we consider the following two-level finite volume methods.

5.1 Simple two-level finite volume approximation

Algorithm:

Step 1 Solve the Navier-Stokes problem on a coarse mesh, i.e., find $(\widetilde{u}_H^v,\widetilde{p}_H^v)\in(X_H,M_H)$, such that for all $(v_H,q_H)\in(X_H,M_H)$,

$$\widetilde{B}_h((\widetilde{u}_H^v,\widetilde{p}_H^v),(v_H,q_H))+b(\widetilde{u}_H^v,\widetilde{u}_H^v,\Gamma_H v_H)=(f,\Gamma_H v_H). \tag{5.1}$$

Step 2 Solve the Stokes problem on a fine mesh, namely, find $(\widetilde{u}_h^v,\widetilde{p}_h^v)\in(X_h,M_h)$ such that for all $(v_h,q_h)\in(X_h,M_h)$,

$$\widetilde{B}_h((\widetilde{u}_h^v,\widetilde{p}_h^v),(v_h,q_h))+b(\widetilde{u}_H^v,\widetilde{u}_H^v,\Gamma_h v_h)=(f,\Gamma_h v_h). \tag{5.2}$$

Next, we denote $(u_h-\widetilde{u}_h^v,p_h-\widetilde{p}_h^v)=(e_h,\eta_h)$ and study the convergence of $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ to (u,p) in some norm. To do this, we subtract (3.8) from (5.2) and obtain the following error equation for any $(v_h,q_h)\in(X_h,M_h)$,

$$
\begin{aligned}
&\widetilde{B}_h((e_h,\eta_h),(v_h,q_h))+b(u_h-\widetilde{u}_H^v,u_h,v_h)+b(\widetilde{u}_H^v,u_h-\widetilde{u}_H^v,v_h)+b(\widetilde{u}_H^v,\widetilde{u}_H^v,v_h-\Gamma_h v_h)\\
=&(f,v_h-\Gamma_h v_h).
\end{aligned}
\tag{5.3}
$$

Theorem 5.1. *Under the conditions of Theorem* 2.1 *and Theorems* 4.1, 4.2 *for H and h, the Simple two-level stabilized finite volume solution $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ satisfies the following error estimates*

$$
\|\nabla(u-\widetilde{u}_h^v)\|_0+\|p-\widetilde{p}_h^v\|_0\le C(h+H^2).
$$

Proof. Taking $(v_h,q_h)=(e_h,\eta_h)$ in (5.3) and using (3.7), we obtain that

$$
\begin{aligned}
&\nu\|\nabla e_h\|_0^2+b(u_h-\widetilde{u}_H^v,u_h,e_h)+b(\widetilde{u}_H^v,u_h-\widetilde{u}_H^v,e_h)+b(\widetilde{u}_H^v,\widetilde{u}_H^v,e_h-\Gamma_h e_h)\\
\le&(f,e_h-\Gamma_h e_h).
\end{aligned}
\tag{5.4}
$$

By applying the definition of $b(\cdot,\cdot,\cdot)$, with the help of (3.2), (3.3), (3.6), Lemmas 3.1, 4.1 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&|b(u_h-\widetilde{u}_H^v,u_h,e_h)|\\
\le&\frac{C}{2}\|A_h u_h\|_0^{\frac12}\|\nabla u_h\|_0^{\frac12}\|u_h-\widetilde{u}_H^v\|_0\|e_h\|_0^{\frac12}\|\nabla e_h\|_0^{\frac12}+\frac{C}{2}\|A_h u_h\|_0^{\frac12}\|\nabla u_h\|_0^{\frac12}\|u_h-\widetilde{u}_H^v\|_0\|\nabla e_h\|_0\\
\le&\frac{C(C_1^{1/2}+1)}{2}\|A_h u_h\|_0^{\frac12}\|\nabla u_h\|_0^{\frac12}\|u_h-\widetilde{u}_H^v\|_0\|\nabla e_h\|_0,\\
&|b(\widetilde{u}_H^v,u_h-\widetilde{u}_H^v,e_h)|\le\frac{C(C_1^{1/2}+1)}{2}\|A_h\widetilde{u}_H^v\|_0^{\frac12}\|\nabla\widetilde{u}_H^v\|_0^{\frac12}\|u_h-\widetilde{u}_H^v\|_0\|\nabla e_h\|_0,\\
&|(f,e_h-\Gamma_h e_h)|\le C_6 h\|f\|_0\|\nabla e_h\|_0,\\
&|b(\widetilde{u}_H^v,\widetilde{u}_H^v,e_h-\Gamma_h e_h)|\\
\le&\Big|((\widetilde{u}_H^v-\Pi_H\widetilde{u}_H^v)\cdot\nabla)\widetilde{u}_H^v+\frac12\mathrm{div}\widetilde{u}_H^v(\widetilde{u}_H^v-\Pi_H\widetilde{u}_H^v),e_h-\Gamma_h e_h\Big|\\
\le&\Big\{\|\nabla\widetilde{u}_H^v\|_{L^\infty}\|\widetilde{u}_H^v-\Pi_H\widetilde{u}_H^v\|_0+\frac{\sqrt2}{2}\|\nabla\widetilde{u}_H^v\|_{L^\infty}\|\widetilde{u}_H^v-\Pi_H\widetilde{u}_H^v\|_0\Big\}\times\|e_h-\Gamma_h e_h\|_0\\
\le&2CC_5C_6|\log H|^{\frac12}Hh\|\nabla\widetilde{u}_H^v\|_0\|A_h\widetilde{u}_H^v\|_0\|\nabla e^n\|_0,
\end{aligned}
$$

which, together (3.10), (3.11), (5.4) with Theorem 4.2 and triangular inequality, under the conditions of (4.2), (4.3) gives

$$
\begin{aligned}
\nu\|\nabla e_h\|_0\le&\Big(C_6\|f\|_0+2CC_5C_6|\log H|^{\frac12}H\|\nabla\widetilde{u}_H^v\|_0\|A_h\widetilde{u}_H^v\|_0\Big)h\\
&+\frac{C(C_1^{\frac12}+1)}{2}\Big(\|A_h\widetilde{u}_H^v\|_0^{\frac12}\|\nabla\widetilde{u}_H^v\|_0^{\frac12}+\|A_h u_h\|_0^{\frac12}\|\nabla u_h\|_0^{\frac12}\Big)\|u_h-\widetilde{u}_H^v\|_0\\
\le&C(h+H^2).
\end{aligned}
\tag{5.5}
$$

Moreover, by using (3.10), (3.11), triangular inequality, Lemmas 3.1, 4.3 and Theorems 4.1, 4.2, we arrive at

$$\begin{aligned}\beta_2\|\eta_h\|_0\le&\sup_{0\neq(v_h,q_h)\in(X_h,M_h)}\frac{|\widetilde{B}_h((e_h,\eta_h);(v_h,q_h))|}{\|v_h\|_1+\|q_h\|_0}\\ \le&\frac{C(C_1^{1/2}+1)}{2}\Big(\|A_h\widetilde{u}_H^v\|_0^{\frac12}\|\nabla\widetilde{u}_H^v\|_0^{\frac12}+\|A_hu_h\|_0^{\frac12}\|\nabla u_h\|_0^{\frac12}\Big)\|u_h-\widetilde{u}_H^v\|_0\\ &+\Big(C_6\|f\|_0+2CC_5C_6|\log H|^{\frac12}H\|\nabla\widetilde{u}_H^v\|_0\|A_h\widetilde{u}_H^v\|_0\Big)h\\ \le&C(h+H^2),\end{aligned}$$

which, together with (3.12) and (5.5), we have finished the proof. ☐

5.2 Newton two-level finite volume approximation

Algorithm:

Step 1 Solve the Navier-Stokes problem on a coarse mesh, i.e., find $(\widetilde{u}_H^v,\widetilde{p}_H^v)\in(X_H,M_H)$ by (5.1).

Step 2 Solve the general Stokes problem on a fine mesh, i.e., apply one Newton step to seek $(\widetilde{u}_h^v,\widetilde{p}_h^v)\in(X_h,M_h)$ such that for all $(v_h,q_h)\in(X_h,M_h)$,

$$\begin{aligned}&\widetilde{B}_h((\widetilde{u}_h^v,\widetilde{p}_h^v),(v_h,q_h))+b(\widetilde{u}_h^v,\widetilde{u}_H^v,\Gamma_hv_h)+b(\widetilde{u}_H^v,\widetilde{u}_h^v,\Gamma_hv_h)\\ =&b(\widetilde{u}_H^v,\widetilde{u}_H^v,\Gamma_hv_h)+(f,\Gamma_hv_h).\end{aligned}\tag{5.6}$$

Next, we will study the convergence of the Newton two-level finite volume solution $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ in some norm. We see from (3.8) and (5.6) that (e_h,η_h) satisfies for all $(v_h,q_h)\in(X_h,M_h)$,

$$\begin{aligned}&\widetilde{B}_h((e_h,\eta_h),(v_h,q_h))+b(e_h,\widetilde{u}_H^v,v_h)+b(\widetilde{u}_H^v,e_h,v_h)+b(u_h-\widetilde{u}_H^v,u_h-\widetilde{u}_H^v,v_h)\\ &\quad+b(\widetilde{u}_h^v,\widetilde{u}_H^v,v_h-\Gamma_hv_h)+b(\widetilde{u}_H^v,\widetilde{u}_h^v-\widetilde{u}_H^v,v_h-\Gamma_hv_h)\\ =&(f,v_h-\Gamma_hv_h).\end{aligned}\tag{5.7}$$

Theorem 5.2. *Under the assumptions of Theorems* 2.1, 4.1, 4.2 *for H and h, and if*

$$\frac{2C_1C_5C_7}{\nu^2}\|f\|_0\le\frac14,$$

then the Newton two-level stabilized finite volume solution $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ *satisfies the following error estimates*

$$\|\nabla(u-\widetilde{u}_h^v)\|_0+\|p-\widetilde{p}_h^v\|_0\le C(h+|\log h|^{\frac12}H^3).$$

Proof. Choosing $(v_h,q_h)=(e_h,\eta_h)$ in (5.7) and using Lemma 3.1, we have

$$
\begin{aligned}
&\nu\|e_h\|_0^2+b(e_h,\widetilde{u}_H^v,e_h)+b(u_h-\widetilde{u}_H^v,u_h-\widetilde{u}_H^v,e_h)\\
&\quad+b(\widetilde{u}_h^v,\widetilde{u}_H^v,e_h-\Gamma_he_h)+b(\widetilde{u}_H^v,\widetilde{u}_h^v-\widetilde{u}_H^v,e_h-\Gamma_he_h)\\
\le&(f,e_h-\Gamma_he_h).
\end{aligned}
\tag{5.8}
$$

Thanks to the definition of $b(\cdot,\cdot,\cdot)$, Eq. (3.2), Lemmas 3.1, 4.1 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&|b(e_h,\widetilde{u}_H^v,e_h)|\le C_3\|\nabla\widetilde{u}_H^v\|_0\|\nabla e_h\|_0^2\le\frac{2C_1C_3C_7}{\nu}\|f\|_0\|\nabla e_h\|_0^2,\\
&|b(u_h-\widetilde{u}_H^v,u_h-\widetilde{u}_H^v,e_h)|\le C_5|\log h|^{\frac12}\|\nabla(u_h-\widetilde{u}_H^v)\|_0\|u_h-\widetilde{u}_H^v\|_0\|\nabla e_h\|_0\\
&\qquad\qquad\qquad\qquad\qquad\quad\le C_5|\log h|^{\frac12}H^3\|\nabla e_h\|_0,\\
&|(f,e_h-\Gamma_he_h)|\le C_6h\|f\|_0\|\nabla e_h\|_0,\\
&|b(\widetilde{u}_h^v,\widetilde{u}_H^v,e_h-\Gamma_he_h)|\\
\le&|b(\widetilde{u}_h^v-u_h,\widetilde{u}_H^v,e_h-\Gamma_he_h)|+|b(u_h-u,\widetilde{u}_H^v,e_h-\Gamma_he_h)|+|b(u,\widetilde{u}_H^v,e_h-\Gamma_he_h)|\\
\le&C_5|\log h|^{\frac12}h\|\nabla\widetilde{u}_H^v\|_0\|\nabla e_h\|_0^2+C_5|\log h|^{\frac12}h^2\|\nabla\widetilde{u}_H^v\|_0\|\nabla e_h\|_0+C_3h\|u\|_2\|\nabla\widetilde{u}_H^v\|_0\|\nabla e_h\|_0\\
\le&\frac{2C_1C_5C_7}{\nu}\|f\|_0\|\nabla e_h\|_0^2+|\log h|^{\frac12}h^2\frac{2C_1C_5C_7}{\nu}\|f\|_0\|\nabla e_h\|_0+h\frac{2C_1C_3C_7}{\nu}\|f\|_0\|u\|_2\|\nabla e_h\|_0,\\
&|b(\widetilde{u}_H^v,\widetilde{u}_h^v-\widetilde{u}_H^v,e_h-\Gamma_he_h)|\\
\le&|b(\widetilde{u}_H^v,e_h,e_h-\Gamma_he_h)|+|b(\widetilde{u}_H^v-u,u_h-\widetilde{u}_H^v,e_h-\Gamma_he_h)|+|b(u,u_h-\widetilde{u}_H^v,e_h-\Gamma_he_h)|\\
\le&C_3\|\nabla\widetilde{u}_H^v\|_0\|\nabla e_h\|_0\|\nabla(e_h-\Gamma_he_h)\|_0+C_5|\log h|^{\frac12}\|\nabla(u-\widetilde{u}_H^v)\|_0\|\nabla(u_h-\widetilde{u}_H^v)\|_0\|e_h-\Gamma_he_h\|_0\\
&\quad+C_3\|u\|_2\|\nabla(u_h-\widetilde{u}_H^v)\|_0\|e_h-\Gamma_he_h\|_0\\
\le&\frac{2C_1C_3C_7}{\nu}\|f\|_0\|\nabla e_h\|_0^2+C_5|\log h|^{\frac12}hH^2\|\nabla e_h\|_0+C_3Hh\|u\|_2\|\nabla e_h\|_0\\
\le&\frac{2C_1C_3C_7}{\nu}\|f\|_0\|\nabla e_h\|_0^2+C_5|\log h|^{\frac12}H^3\|\nabla e_h\|_0+C_3h\|u\|_2\|\nabla e_h\|_0,
\end{aligned}
$$

which, together with (5.8), yields

$$
\begin{aligned}
&\nu\Big(1-\frac{2C_1(2C_3+C_5)C_7}{\nu^2}\|f\|_0\Big)\|\nabla e_h\|_0\\
\le&2C_5|\log h|^{\frac12}H^3+\Big(C_6\|f\|_0+|\log h|^{\frac12}h\frac{2C_1C_5C_7}{\nu}\|f\|_0+\frac{2C_1C_3C_7}{\nu}\|f\|_0\|u\|_2+C_3\|u\|_2\Big)h.
\end{aligned}
$$

Under the conditions of (4.2), (4.3), with Theorems 4.1, 4.2, we have

$$
\|\nabla e\|_0\le C(|\log h|^{\frac12}H^3+h). \tag{5.9}
$$

Moreover, by applying (3.10), (3.11), Lemmas 3.1, 4.3 and Theorems 4.1, 4.2, we arrive at

$$\begin{aligned}\beta_2\|\eta_h\|_0 \le & \sup_{0\neq(v_h,q_h)\in(X_h,M_h)} \frac{|\widetilde{B}_h((e_h,\eta_h);(v_h,q_h))|}{\|v_h\|_1+\|q_h\|_0} \\ \le & \frac{2C_1(2C_3+C_5)C_7}{\nu}\|f\|_0\|\nabla e_h\|_0 + C|\log h|^{\frac{1}{2}}H^3 \\ & + \Big(C_6\|f\|_0+|\log h|^{\frac{1}{2}}h\frac{2C_1C_5C_7}{\nu}\|f\|_0+\frac{2C_1C_3C_7}{\nu}\|f\|_0\|u\|_2+C_3\|u\|_2\Big)h \\ \le & C(|\log h|^{\frac{1}{2}}H^3+h),\end{aligned}$$

which together with (5.9), we have completed the proof. □

5.3 Oseen two-level finite volume approximation

Algorithm:

Step 1 Solve the Navier-Stokes problem on a coarse mesh, i.e., fine $(\widetilde{u}_H^v,\widetilde{p}_H^v)\in(X_H,M_H)$ by (5.1).

Step 2 Solve the generalized Stokes problem on a fine mesh, i.e., apply one Oseen step to find $(\widetilde{u}_h^v,\widetilde{p}_h^v)\in(X_h,M_h)$ such that for all $(v_h,q_h)\in(X_h,M_h)$

$$\widetilde{B}_h((\widetilde{u}_h^v,\widetilde{p}_h^v),(v_h,q_h))++b(\widetilde{u}_H^v,\widetilde{u}_h^v,\Gamma_h v_h)=(f,\Gamma_h v_h). \tag{5.10}$$

Next, we will study the convergence of the Oseen two-level finite volume solution $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ in some norm. We see from (3.8) and (5.6) that (e_h,η_h) satisfies for all $(v_h,q_h)\in(X_h,M_h)$,

$$\begin{aligned}&\widetilde{B}_h((e_h,\eta_h),(v_h,q_h))+b(u_h-\widetilde{u}_H^v,u_h,v_h)+b(\widetilde{u}_H^v,u_h-\widetilde{u}_h^v,v_h)+b(\widetilde{u}_H^v,\widetilde{u}_h^v,v_h-\Gamma_h v_h)\\ =&(f,v_h-\Gamma_h v_h).\end{aligned} \tag{5.11}$$

Theorem 5.3. *Under the conditions of Theorems* 2.1, 4.1, 4.2 *for H and h, the Oseen two-level stabilized finite volume solution $(\widetilde{u}_h^v,\widetilde{p}_h^v)$ satisfies the following error estimates*

$$\|\nabla(u-\widetilde{u}_h^v)\|_0+\|p-\widetilde{p}_h^v\|_0\le C(h+H^2).$$

Proof. Choosing $(v_h,q_h)=(e_h,\eta_h)$ in (5.11) and using Lemma 3.1, we have

$$\nu\|e_h\|_0^2+b(u_h-\widetilde{u}_H^v,u_h,e_h)+b(\widetilde{u}_H^v,\widetilde{u}_h^v,e_h-\Gamma_h e_h)\le(f,e_h-\Gamma_h e_h). \tag{5.12}$$

Combining with the definition of $b(\cdot,\cdot,\cdot)$, (3.10), (3.11), Lemmas 3.1, 4.1, Theorems 4.1, 4.2

and the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
&|b(u_h-\widetilde{u}_H^v,u_h,e_h)|\leq CC_1\|u_h-\widetilde{u}_H^v\|_0\|A_hu_h\|_0\|\nabla e_h\|_0\\
&\qquad\qquad\qquad\qquad\leq CC_1\Big(1+\frac{C^2C_1^4}{2\nu^4}\|f\|_0^2\Big)\|f\|_0H^2\|\nabla e_h\|_0,\\
&|(f,e_h-\Gamma_he_h)|\leq C_6h\|f\|_0\|\nabla e_h\|_0,\\
&|b(\widetilde{u}_H^v,\widetilde{u}_h^v,e_h-\Gamma_he_h)|\\
\leq&\Big|\Big(((\widetilde{u}_H^v-\Pi_H\widetilde{u}_H^v)\cdot\nabla)\widetilde{u}_h^v+\frac{1}{2}\mathrm{div}\widetilde{u}_H^v(\widetilde{u}_h^v-\Pi_h\widetilde{u}_h^v),e_h-\Gamma_he_h\Big)\Big|\\
\leq&CC_1^{\frac{1}{2}}C_6Hh\|\nabla\widetilde{u}_H^v\|_0\|\nabla\widetilde{u}_h^v\|_0^{\frac{1}{2}}h^{-\frac{1}{2}}\|A_h\widetilde{u}_h^v\|_0^{\frac{1}{2}}\|\nabla e_h\|_0\\
&+\frac{\sqrt{2}}{2}CC_5C_6|\log H|^{\frac{1}{2}}h^2\|\nabla\widetilde{u}_h^v\|_0\|A_h\widetilde{u}_H^v\|_0\|\nabla e_h\|_0\\
\leq&CC_5C_6|\log H|^{\frac{1}{2}}h\|\nabla\widetilde{u}_h^v\|_0\|A_H\widetilde{u}_H^v\|_0\|\nabla e_h\|_0h\\
&+CC_1^{\frac{1}{2}}C_6\|\nabla\widetilde{u}_H^v\|_0\|\nabla\widetilde{u}_h^v\|_0^{\frac{1}{2}}\|A_h\widetilde{u}_h^v\|_0^{\frac{1}{2}}\|\nabla e_h\|_0Hh^{\frac{1}{2}},
\end{aligned}
$$

which, together with (5.12) and Cauchy inequality yields

$$
\begin{aligned}
\nu\|\nabla e_h\|_0\leq&CC_1\Big(1+\frac{C^2C_1^4}{2\nu^4}\|f\|_0^2\Big)\|f\|_0H^2+CC_5C_6|\log H|^{\frac{1}{2}}h\|\nabla\widetilde{u}_h^v\|_0\|A_H\widetilde{u}_H^v\|_0h\\
&+C_6\|f\|_0h+CC_1^{\frac{1}{2}}C_6\|\nabla\widetilde{u}_H^v\|_0\|\nabla\widetilde{u}_h^v\|_0^{\frac{1}{2}}\|A_h\widetilde{u}_h^v\|_0^{\frac{1}{2}}Hh^{\frac{1}{2}}\\
\leq&C(H^2+h).
\end{aligned}\tag{5.13}
$$

Moreover, thanks to (3.10), (3.11), Lemmas 3.1, 4.1, 4.3 and Theorems 4.1, 4.2, we arrive at

$$
\begin{aligned}
\beta_2\|\eta_h\|_0\leq&\sup_{0\neq(v_h,q_h)\in(X_h,M_h)}\frac{|\widetilde{B}_h((e_h,\eta_h);(v_h,q_h))|}{\|v_h\|_1+\|q_h\|_0}\\
\leq&CC_1\big(\|A_hu_h\|_0\|u_h-\widetilde{u}_H^v\|_0+\|A_H\widetilde{u}_H^v\|_0\|u_h-\widetilde{u}_h^v\|_0\big)+C_6\|f\|_0h\\
&+CC_5C_6|\log H|^{\frac{1}{2}}h\|\nabla\widetilde{u}_h^v\|_0\|A_H\widetilde{u}_H^v\|_0h+CC_1^{\frac{1}{2}}C_6\|\nabla\widetilde{u}_H^v\|_0\|\nabla\widetilde{u}_h^v\|_0^{\frac{1}{2}}\|A_h\widetilde{u}_h^v\|_0^{\frac{1}{2}}Hh^{\frac{1}{2}}\\
\leq&C(H^2+h).
\end{aligned}
$$

Combining (3.12), (5.13) with triangular inequality yields the desired result. □

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