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Two-Level Newton Iteration Methods for Navier-Stokes Type Variational Inequality Problem

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Abstract. This paper deals with the two-level Newton iteration method based on the pressure projection stabilized finite element approximation to solve the numerical solution of the Navier-Stokes type variational inequality problem. We solve a small Navier-Stokes problem on the coarse mesh with mesh size H and solve a large linearized Navier-Stokes problem on the fine mesh with mesh size h. The error estimates derived show that if we choose $h=\mathcal{O}(|\log h|^{1/2}H^3)$, then the two-level method we provide has the same H^1 and L^2 convergence orders of the velocity and the pressure as the one-level stabilized method. However, the L^2 convergence order of the velocity is not consistent with that of one-level stabilized method. Finally, we give the numerical results to support the theoretical analysis.

AMS subject classifications: 65N30

Key words: Navier-Stokes equations, nonlinear slip boundary conditions, variational inequality problem, stabilized finite element, two-level methods.

1 Introduction

In this paper, we deal with the steady Navier-Stokes equations:

$$\begin{cases} -\mu\Delta u+(u\cdot\nabla)u+\nabla p=f, & \text{in } \Omega, \\ \text{div}u=0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded and convex domain. $\mu>0$ denotes the kinetic viscous coefficient, u and p denote the velocity and the pressure, respectively. f denotes the external

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body force. $\mathrm{div}u=0$ implies that the fluid is incompressible. We suppose that the boundary $\partial\Omega$ of Ω is composed of two parts Γ and S which satisfy $\mathrm{meas}(\Gamma)\neq 0$, $\mathrm{meas}(S)\neq 0$, $\Gamma\cap S=\emptyset$, $\overline{\Gamma\cup S}=\partial\Omega$. Unlike the usual whole Dirichlet boundary conditions, we consider the following the nonlinear slip boundary conditions of friction type:

$$\begin{cases} u=0, & \text{on } \Gamma, \\ u_n=0, \quad -\sigma_\tau(u)\in g\partial|u_\tau|, & \text{on } S, \end{cases} \tag{1.2}$$

where $g\geq 0$ is a scalar function. $u_n=u\cdot n$ and $u_\tau=u\cdot\tau$ are the normal and tangential components of the velocity, where n and τ stand for the unit vector of the external normal and the tangential vector to S. $\sigma_\tau(u)=\sigma\cdot\tau$, independent of p, is the tangential components of the stress vector σ defined by $\sigma_i=\sigma_i(u,p)=(\mu e_{ij}(u)-p\delta_{ij})n_j$, where $e_{ij}(u)=\partial u_i/\partial x^j+\partial u_j/\partial x^i$, $i,j=1,2$. The set $\partial|u_\tau|$ denotes a subdifferential of the absolute value function at the point u_τ, which is defined by

$$\partial|u_\tau|=\big\{b\in\mathbb{R}:|h|-|u_\tau|\geq b\cdot(h-u_\tau),\ \forall\, h\in\mathbb{R}\big\}.$$

The Navier-Stokes equations with nonlinear slip boundary conditions of friction type is firstly introduced by Fujita in [1] and appears in the modeling of blood flow in a vein of an arterial sclerosis patient. There have some theoretical results, especially for the well-posedness analysis of the Stokes problem. We refer to Fujita [2–4], Saito [5], Li [6] and the references cited therein. Some scholars have focused on the numerical methods. For example, Suito and his collaborates have applied the boundary conditions (1.2) to some flow phenomena by the finite difference methods in [7–9], such as the oil flow over or beneath sand layers and the blood flow in the thoracic aorta. Ayadi and his collaborates in [10] studied the finite element approximation for the Stokes problem, where they use the P_1b-P_1 element and derived the error estimates in virtue of the Lagrange multiplier method. Kwshiwabara in [11] used the Taylor-Hood element and obtained the optimal error estimates for the Stokes problem. Recently, we in [12] applied the pressure projection stabilized finite element method to the steady Navier-Stokes problem and constructed the simple and the Oseen two-level iteration schemes. We showed that if the coarse mesh size H and the fine mesh size h satisfy $h=\mathcal{O}(H^2)$, then the error estimates indicate the simple or Oseen two-level methods will provide the same order of the approximation as the usual one-level stabilized finite element method [13]. Much research works have been done about the finite element analysis the variational inequality problems associated with the Navier-Stokes equations. We refer to the following works [14–16] and the references cited therein.

In this paper, based on the Newton iteration scheme [17–19], we continue to study the two-level finite element methods for the Navier-Stokes equations with the boundary conditions (1.2). The main idea is solving a small Navier-Stokes type variational inequality problem on the coarse mesh with mesh size H and solving a large linearized Navier-Stokes type variational inequality problem on the fine mesh with mesh size h in virtue of the Newton iteration scheme. Denote the approximation solution on the fine mesh by (u^h,p^h). We prove the following error estimate:

$$\begin{cases} \|u-u^h\|_{H^1}+\|p-p^h\|_{L^2}\leq c\big(h+|\log h|^{\frac{1}{2}}H^3\big), \\ \|u-u^h\|\leq c\big(h^2+h|\log h|^{\frac{1}{2}}H^3+H^{4-\varepsilon}\big), \end{cases} \tag{1.3}$$

where $0<\varepsilon\leq 1$, $c>0$ is independent of h, (u,p) is the solution of the problem (1.1)-(1.2). Hence, if we choose $h=\mathcal{O}(|\log h|^{1/2}H^3)$, then the Newton two-level method we provide is of the same H^1 and L^2 convergence orders of the velocity and the pressure as the one-level stabilized finite element method [13]. However, the L^2 convergence order of the velocity is not consistent with that of one-level stabilized method.

This paper is organized as follows. In Section 2, we will give the variational formulation of the problem (1.1)-(1.2) and recall some theoretical results. In Section 3, we will describe the pressure projection stabilized finite element approximation. In Section 4, we will give the two-level Newton iteration scheme and show the error estimates (1.3). In Section 5, the numerical experiments are provided to support the theoretical results.

2 Navier-Stokes equations with nonlinear slip boundary conditions

First, we introduce some function spaces used in this paper.

$$\begin{aligned} &\mathcal{V}=\big\{u\in H^1(\Omega)^2,\, u|_\Gamma=0,\, u\cdot n|_S=0\big\}, \qquad \mathcal{V}_0=H_0^1(\Omega)^2, \\ &\mathcal{V}_\sigma=\big\{u\in V,\, \mathrm{div}u=0\big\}, \qquad \mathcal{H}=\big\{u\in L^2(\Omega)^2,\, \mathrm{div}u=0,\, u\cdot n|_{\partial\Omega}=0\big\}, \\ &\mathcal{M}=L_0^2(\Omega)=\Big\{q\in L^2(\Omega),\, \int_\Omega q\mathrm{d}x=0\Big\}. \end{aligned}$$

Denote the inner product and the norm in $L^2(\Omega)$ or $L^2(\Omega)^2$ by $(\cdot,\cdot)$ and $\|\cdot\|$. Let $\|\cdot\|_k$ denote the usually Sobolev norm in $H^k(\Omega)^2$. Then we can equip the inner product and the norm in $\mathcal{V}$ by $(\nabla\cdot,\nabla\cdot)$ and $\|\cdot\|_V=\|\nabla\cdot\|$, because $\|\nabla v\|$ is equivalent to $\|v\|_1$ for all $v\in\mathcal{V}$ in terms of the Poincare's inequality.

Next, we introduce the following bilinear and trilinear forms:

$$\begin{aligned} &a(u,v)=\mu\int_\Omega \nabla u\cdot\nabla v dx, &&\forall u,v\in\mathcal{V}, \\ &b(u,v,w)=\int_\Omega u\cdot\nabla v\cdot w dx, &&\forall u,v,w\in\mathcal{V}, \\ &d(v,p)=\int_\Omega p\mathrm{div}v dx, &&\forall v\in\mathcal{V},\ \ p\in\mathcal{M}. \end{aligned}$$

It is obvious that $a(v,v)=\mu\|v\|_V^2$ for all $v\in\mathcal{V}$. Moreover, if $\mathrm{div}u=0$, the trilinear term

$b(\cdot,\cdot,\cdot):\mathcal{V}\times\mathcal{V}\times\mathcal{V}\rightarrow\mathbb{R}$ satisfies

$$\begin{aligned} b(u,v,w)&=((u\cdot\nabla)v,w)+\frac{1}{2}((\mathrm{div}u)v,w)\\ &=\frac{1}{2}((u\cdot\nabla)v,w)-\frac{1}{2}((u\cdot\nabla)w,v), \quad \forall u\in\mathcal{V}_\sigma,\ v,w\in\mathcal{V}. \end{aligned}$$

Thus $b(\cdot,\cdot,\cdot):\mathcal{V}_\sigma\times\mathcal{V}\times\mathcal{V}\rightarrow\mathbb{R}$ satisfies the antisymmetric property, i.e.,

$$b(u,v,w)=-b(u,w,v), \quad \forall u\in\mathcal{V}_\sigma,\ v,w\in\mathcal{V}.$$

Denote

$$N=\sup_{u,v,w\in V}\frac{b(u,v,w)}{\|u\|_V\|v\|_V\|w\|_V}.$$

Then there holds

$$b(u,v,w)\le N\|u\|_V\|v\|_V\|w\|_V, \quad \forall u,v,w\in\mathcal{V}.$$

Given $f\in L^2(\Omega)^2$ and $g\in L^2(S)$ with $g\ge 0$ on S, the weak variational formulation of (1.1)-(1.2) is the following variational inequality problem:

$$\begin{cases} \text{find } (u,p)\in\mathcal{V}\times\mathcal{M} \text{ such that}\\ a(u,v-u)+b(u,u,v-u)+j(v_\tau)\\ \quad -j(u_\tau)-d(v-u,p)\ge(f,v-u), & \forall v\in\mathcal{V},\\ d(u,q)=0, & \forall q\in\mathcal{M}, \end{cases} \tag{2.1}$$

where $j(\eta)=\int_S g|\eta|ds$. We call (2.1) the Navier-Stokes type variational inequality problem. Since Saito in [5] has shown that the bilinear form $d(\cdot,\cdot):\mathcal{V}\times\mathcal{M}\rightarrow\mathbb{R}$ satisfies the inf-sup condition, then using the classical argument, the variational inequality problem (2.1) is equivalent to

$$\begin{cases} \text{find } u\in\mathcal{V}_\sigma \text{ such that}\\ a(u,v-u)+b(u,u,v-u)+j(v_\tau)-j(u_\tau)\ge(f,v-u), \quad \forall\, v\in\mathcal{V}_\sigma. \end{cases} \tag{2.2}$$

About the existence and the uniqueness of the solution u, Li in [13] has established in terms of the contraction mapping principle. Here, we only recall this result.

Theorem 2.1. *Suppose that the uniqueness condition*

$$\frac{4\kappa_1 N(\|f\|+\|g\|_{L^2(S)})}{\mu^2}<1 \tag{2.3}$$

holds, then the variational inequality problem (2.2) *admits a unique solution* $u\in\mathcal{K}_\sigma$, *where* $\mathcal{K}_\sigma=\{v\in\mathcal{V}_\sigma:\ \|v\|_V\le 2\kappa_1(\|f\|+\|g\|_{L^2(S)})/\mu\}$ *and* $\kappa_1>0$ *satisfies*

$$|(f,v)-j(v_\tau)|\le\kappa_1(\|f\|+\|g\|_{L^2(S)})\|v\|_V, \quad \forall v\in\mathcal{V}_\sigma.$$

3 Pressure projection stabilized finite element approximation

Pressure projection stabilized method is introduced by Bochev and his collaborates in [20] and is based on the low-order conforming finite element, such as P_1-P_1 element or P_1-P_0 element. The stable condition is achieved by projecting the P_0 (or P_1) finite element space for the pressure to the P_1 (or P_0) finite element space. Moreover, This stabilized method is unconditional stable and has been applied to the Navier-Stokes equations with whole Dirichlet boundary conditions. We refer to the following works [21–23] and the references cited therein. In this paper, we will extent the pressure projection stabilized method combining the two-level Newton type scheme to solve Navier-Stokes type variational inequality problem (2.1).

Let $\mathcal{T}_h$ be a family of regular triangular partition of Ω into triangles not greater than $0<h<1$. For every $K\in\mathcal{T}_h$, denote the space of the polynomials on K of degree at most r. Define the finite element space of $\mathcal{V}$ and $\mathcal{M}$ by

$$\mathcal{V}_h=\{v\in\mathcal{V}:\ v|_K\in P_1(K),\ \ \forall K\in\mathcal{T}_h\},$$

and

$$\mathcal{M}_h=\{q\in\mathcal{M}:\ q|_K\in P_1(K),\ \ \forall K\in\mathcal{T}_h\}.$$

Then the pressure projection finite element approximation solution $(u_h,p_h)\in\mathcal{V}_h\times\mathcal{M}_h$ of (2.1) satisfies the following discrete variational inequality problem:

$$\begin{cases} a(u_h,v_h-u_h)+b(u_h,u_h,v_h-u_h)+j(v_{h\tau}) \\ \quad -j(u_{h\tau})-d(v_h-u_h,p_h)\geq(f,v_h-u_h), & \forall v_h\in\mathcal{V}_h, \\ d(u_h,q_h)+G(p_h,q_h)=0, & \forall q_h\in\mathcal{M}_h, \end{cases} \tag{3.1}$$

where the stabilized term $G(p,q)$ is defined by

$$G(p,q)=(p-\Pi p,q-\Pi q),\quad \forall p,q\in\mathcal{M}.$$

Here the operator $\Pi:\mathcal{M}\to P_0$ has a piecewise constant range and satisfies

$$\|p-\Pi p\|\leq ch\|p\|_1,\quad \forall p\in H^1(\Omega), \tag{3.2}$$

where $c>0$ is independent of h.

Define the generalized bilinear form $\mathcal{B}:(\mathcal{V},\mathcal{M})\times(\mathcal{V},\mathcal{M})\to\mathbb{R}$ by

$$\mathcal{B}(u,p;v,q)=a(u,v)-d(v,p)+d(u,q).$$

Denote

$$\mathcal{B}_h(u_h,p_h;v_h,q_h)=\mathcal{B}(u_h,p_h;v_h,q_h)+G(p_h,q_h).$$

Then the discrete problem (3.1) is rewritten as follows:

$$\mathcal{B}_h(u_h,p_h;v_h-u_h,q_h-p_h)+b(u_h,u_h,v_h-u_h)+j(v_{h\tau})-j(u_{h\tau})\geq(f,v_h-u_h). \tag{3.3}$$

In order to establish the existence and uniqueness of the solution to (3.3), we recall the following stable theorem due to [20,21].

Theorem 3.1. *For all $p\in\mathcal{M}$, suppose that Π is continuous as an operator $\mathcal{M}\to P_0$:*

$$\|\Pi p\|\le c\|p\|,\quad \forall p\in\mathcal{M},$$

then $\mathcal{B}_h$ satisfies the following continuous property:

$$|\mathcal{B}_h(u_h,p_h;v_h,q_h)|\le\beta_1(\|u_h\|_V+\|p_h\|)(\|v_h\|_V+\|q_h\|),\quad \forall(u_h,p_h),(v_h,q_h)\in\mathcal{V}_h\times\mathcal{M}_h,$$

and the weakly coercive property:

$$\beta_2(\|u_h\|_V+\|p_h\|)\le\sup_{(v_h,q_h)\in\widetilde{\mathcal{V}_h}\times\mathcal{M}_h}\frac{\mathcal{B}_h(u_h,p_h;v_h,q_h)}{\|v_h\|_V+\|q_h\|},\quad \forall(u_h,p_h)\in\widetilde{\mathcal{V}_h}\times\mathcal{M}_h,$$

where $\widetilde{\mathcal{V}_h}=\mathcal{V}_h\cap\mathcal{V}_0$, $\beta_1>0$, $\beta_2>0$ are two constants independent of h.

We recall the results about the existence and uniqueness of the solution to the discrete problem (3.3) and the error estimate between u and u_h in [13].

Theorem 3.2. *Suppose that the uniqueness condition* (2.3) *holds, then the discrete problem* (3.3) *admits a unique solution $(u_h,p_h)\in\mathcal{K}_h$, where*

$$\mathcal{K}_h=\Big\{(v_h,q_h)\in\mathcal{V}_h\times\mathcal{M}_h,\ \|v_h\|_V\le\frac{2\kappa_1}{\mu}(\|f\|+\|g\|_{L^2(S)}),\ \|q_h\|\le\frac{\|f\|+\kappa_1(\|f\|+\|g\|_{L^2(S)})}{\beta_2}\Big\}.$$

Theorem 3.3. *Let $(u,p)\in\mathcal{V}\times\mathcal{M}$ and $(u_h,p_h)\in\mathcal{V}_h\times\mathcal{M}_h$ be the solutions of* (2.1) *and* (3.3), *respectively. If (u,p) is sufficiently smooth, then we have the following optimal error estimate*

$$\|u-u_h\|+h\|u-u_h\|_V+h\|p-p_h\|\le ch^2, \tag{3.4}$$

where $c>0$ is independent of h.

4 Two-level Newton iteration scheme

In this section, we will give the two-level Newton iteration scheme to solve the numerical solution of (3.3). Let $\mathcal{T}_H$ and $\mathcal{T}_h$ be the family of the regular triangular partition of Ω into triangles of diameter not great than H and h satisfying $0<h\ll H<1$. The finite element spaces $(\mathcal{V}_H,\mathcal{M}_H)$ and $(\mathcal{V}_h,\mathcal{M}_h)$ associated with the partition $\mathcal{T}_H$ and $\mathcal{T}_h$ are defined as like in Section 3. The suggested two-level Newton iteration scheme is required to solve a small Navier-Stokes type variational inequality problem on the coarse mesh and solve a large linearized Navier-Stokes type variational inequality problem on the fine mesh, which is constructed as follows:

Step 1 Solve $(u_H,p_H)\in\mathcal{V}_h\times\mathcal{M}_h$ such that for all $(v_H,q_H)\in\mathcal{V}_h\times\mathcal{M}_h$ there holds

$$\begin{aligned}&\mathcal{B}_H(u_H,p_H;v_H-u_H,q_H-p_H)+b(u_H,u_H,v_H-u_H)+j(v_{H\tau})-j(u_{H\tau})\\ \ge&(f,v_H-u_H).\end{aligned} \tag{4.1}$$

Step 2 Solve $(u^h,p^h)\in\mathcal{V}_h\times\mathcal{M}_h$ such that for all $(v_h,q_h)\in\mathcal{V}_h\times\mathcal{M}_h$ there holds

$$\begin{aligned}&\mathcal{B}_h(u^h,p^h;v_h-u^h,q_h-p^h)+b(u_H,u^h,v_h-u^h)+b(u^h,u_H,v_h-u^h)+j(v_{h\tau})-j(u^h_\tau)\\ \geq&(f,v_h-u^h)+b(u_H,u_H,v_h-u^h).\end{aligned}\tag{4.2}$$

In terms of Theorem 3.2, the problem (4.1) has a unique solution $(u_H,p_H)\in\mathcal{K}_H$. Moreover, the approximation solution satisfies

$$\|u-u_H\|+h\|u-u_H\|_V+h\|p-p_H\|\leq cH^2.\tag{4.3}$$

About the problem (4.2), from the uniqueness condition (2.3), we have

$$\begin{aligned}&\mathcal{B}_h(u^h,p^h;u^h,p^h)+b(u_H,u^h,u^h)+b(u^h,u_H,u^h)\\ \geq&\mu\|u^h\|_V^2+G(p^h,p^h)-N\|u_H\|_V\|u^h\|_V^2\\ \geq&\mu\|u^h\|_V^2+G(p^h,p^h)-\frac{2N\kappa_1}{\mu}(\|f\|+\|g\|_{L^2(S)})\|u^h\|_V^2\\ \geq&\frac{\mu}{2}\|u^h\|_V^2+G(p^h,p^h).\end{aligned}$$

Then the discrete problem (4.2) exists a unique solution $(u^h,p^h)\in\mathcal{V}_h\times\mathcal{M}_h$ satisfying

$$\|u^h\|_V\leq\frac{2}{\mu}(\|f\|+N\|u_H\|_V^2)<+\infty.\tag{4.4}$$

Define the Galerkin projection operator $(R_h,Q_h):\mathcal{V}\times\mathcal{M}\to\mathcal{V}_h\times\mathcal{M}_h$ by

$$\mathcal{B}_h(R_hu,Q_hp;w_h,q_h)=\mathcal{B}(u,p;w_h,q_h),\quad\forall(w_h,q_h)\in\mathcal{V}_h\times\mathcal{M}_h.$$

Then, according to Theorem 3.1, we obtain

$$\begin{aligned}\beta_2(\|R_hu\|_V+\|Q_hp\|)\leq&\sup_{(w_h,q_h)\in(\widetilde{\mathcal{V}}_h,\mathcal{M}_h)}\frac{\mathcal{B}_h(R_hu,Q_hp;w_h,q_h)}{\|w_h\|_V+\|q_h\|}\\ \leq&\sup_{(w_h,q_h)\in(\widetilde{\mathcal{V}}_h,\mathcal{M}_h)}\frac{a(u,v_h)-d(v_h,p)+d(u,q_h)}{\|w_h\|_V+\|q_h\|}\\ \leq&c(\|u\|_V+\|p\|)<+\infty,\end{aligned}\tag{4.5}$$

where $c>0$ is some positive constant depending on μ.

By using the similar argument in [24], the following approximation property can be obtained.

Lemma 4.1. *For sufficiently smooth (u,p), the projection (R_hu,Q_hp) of (u,p) satisfies*

$$\|u-R_hu\|+h\|u-R_hu\|_V+h\|p-Q_hp\|\leq ch^2,\tag{4.6}$$

where $c>0$ is independent of h.

Proof. Let $(u,p)\in H^2(\Omega)^2\times H^1(\Omega)$. Denote $I_h:H^2(\Omega)^2\cap\mathcal{V}\to\mathcal{V}_h$ and $J_h:H^1(\Omega)\cap\mathcal{M}\to\mathcal{M}_h$ are the standard interpolation operators and satisfy

$$\begin{cases} \|v-I_hv\|+h\|v-I_hv\|_V\le ch^2\|v\|_2, & \forall v\in H^2(\Omega)^2\cap\mathcal{V},\\ \|q-J_hq\|\le ch\|q\|_1, & \forall q\in H^1(\Omega)\cap\mathcal{M}. \end{cases}$$

In views of Theorem 3.1, we have

$$\begin{aligned} &\beta_2(\|I_hu-R_hu\|_V+\|J_hp-Q_hp\|)\\ \le&\sup_{(w_h,q_h)\in(\widetilde{\mathcal{V}}_h,\mathcal{M}_h)}\frac{\mathcal{B}_h(I_hu-R_hu,J_hp-Q_hp;w_h,q_h)}{\|w_h\|_V+\|q_h\|}\\ \le&\sup_{(w_h,q_h)\in(\widetilde{\mathcal{V}}_h,\mathcal{M}_h)}\frac{B_h(I_hu-u,J_hp-p;w_h,q_h)+G(p,q_h)}{\|w_h\|_V+\|q_h\|}\\ \le& c(\|u-I_hu\|_V+\|p-J_hp\|)+ch\|p\|_{H^1}\\ \le& ch(\|u\|_{H^2}+\|p\|_{H^1}). \end{aligned} \tag{4.7}$$

Then from the triangular inequality, we obtain

$$\|u-R_hu\|_V+\|p-Q_hp\|\le ch.$$

Consider the following dual Stokes problem: find $(\Phi,\Psi)\in\mathcal{V}\times M$ such that

$$\mathcal{B}(w,r;\Phi,\Psi)=(w,u-R_hu),\quad \forall(w,r)\in\mathcal{V}\times M. \tag{4.8}$$

Following the regularity results about the Stokes problem, the problem (4.7) admits a solution (Φ,Ψ) satisfying

$$\|\Phi\|_2+\|\Psi\|_1\le c\|u-R_hu\|, \tag{4.9}$$

where $c>0$ depends on μ and Ω.

Let $(\Phi_h,\Psi_h)=(I_h\Phi,J_h\Psi)$. Then setting $w=u-R_hu$ and $r=p-Q_hp$ in (4.8), we have

$$\begin{aligned} \|u-R_hu\|^2=&\mathcal{B}(u-R_hu,p-Q_hp;\Phi,\Psi)\\ =&\mathcal{B}_h(u-R_hu,p-Q_hp;\Phi,\Psi)-G(p-Q_hp;\Psi)\\ =&\mathcal{B}_h(u-R_hu,p-Q_hp;\Phi-\Phi_h,\Psi-\Psi_h)+G(p,\Psi_h)-G(p-Q_hp;\Psi)\\ =&\mathcal{B}_h(u-R_hu,p-Q_hp;\Phi-\Phi_h,\Psi-\Psi_h)\\ &\quad+G(p,\Psi_h-\Psi)+G(p,\Psi)-G(p-Q_hp;\Psi)\\ \le& ch(\|u-R_hu\|_V+\|p-Q_hp\|)(\|\Phi\|_2+\|\Psi\|_1)\\ &\quad+c\|p-\Pi p\|\cdot\|\Psi_h-\Psi\|+c\|p-\Pi p\|\cdot\|\Psi-\Pi\Psi\|\\ &\quad+c\|p-Q_hp\|\cdot\|\Psi-\Pi\Psi\|\\ \le& ch^2(\|\Phi\|_2+\|\Psi\|_1)\le ch^2\|u-R_hu\|, \end{aligned}$$

which implies that

$$\|u-R_hu\|\le ch^2.$$

Thus, the lemma is proved. □

Theorem 4.1. *Suppose that the uniqueness condition* (2.3) *holds. For sufficiently smooth* (u,p)*, if* $|\log h|^{1/2}H^2<1$*, then the two-level Newton iteration solution* (u^h,p^h) *satisfies the following error estimate:*

$$\|u-u^h\|_V+\|p-p^h\|\le c(h+|\log h|^{\frac{1}{2}}H^3), \tag{4.10}$$

where $c>0$ *is independent of* h *and* H.

Proof. For all $(v_h,q_h)\in\mathcal{V}_h\times\mathcal{M}_h$, we have

$$\begin{aligned}
&\mu\|u^h-v_h\|_V^2+G(p^h-q_h,p^h-q_h)\\
=&\mathcal{B}_h(u^h-v_h,p^h-q_h;u^h-v_h,p^h-q_h)\\
=&\mathcal{B}_h(u^h,p^h;u^h-v_h,p^h-q_h)-\mathcal{B}_h(v_h,q_h;u^h-v_h,p^h-q_h)\\
\le&(f,u^h-v_h)+b(u_H,u_H,u^h-v_h)-b(u^h,u_H,u^h-v_h)-b(u_H,u^h,u^h-v_h)\\
&+j(v_{h\tau})-j(u^h_\tau)-\mathcal{B}_h(v_h,q_h;u^h-v_h,p^h-q_h).
\end{aligned}\tag{4.11}$$

Set $v=u^h$ and $v=2u-v_h$ in (2.1). Then we obtain

$$\begin{aligned}
&a(u,u^h-u)+b(u,u,u^h-u)+j(u^h_\tau)-j(u_\tau)-d(u^h-u,p)\ge(f,u^h-u),\\
&a(u,u-v_h)+b(u,u,u-v_h)-d(u-v_h,p)+j(2u_\tau-v_{h\tau})-j(u_\tau)\ge(f,u-v_h).
\end{aligned}$$

Summing the above two inequality yields

$$\begin{aligned}
&a(u,u^h-v_h)+b(u,u,u^h-v_h)+j(2u_\tau-v_{h\tau})-2j(u_\tau)+j(u^h_\tau)-d(u^h-v_h,p)\\
\ge&(f,u^h-v_h).
\end{aligned}\tag{4.12}$$

Substituting (4.12) into (4.11) and noting $d(u,q)=0$ for all $q\in\mathcal{M}$, we obtain

$$\begin{aligned}
&\mu\|u^h-v_h\|_V^2+G(p^h-q_h,p^h-q_h)\\
\le&a(u,u^h-v_h)+b(u,u,u^h-v_h)+j(2u_\tau-v_{h\tau})-2j(u_\tau)-d(u^h-v_h,p)\\
&+b(u_H,u_H,u^h-v_h)-b(u^h,u_H,u^h-v_h)-b(u_H,u^h,u^h-v_h)\\
&+j(v_{h\tau})-\mathcal{B}_h(v_h,q_h;u^h-v_h,p^h-q_h)\\
\le&a(u,u^h-v_h)+b(u,u,u^h-v_h)+j(2u_\tau-v_{h\tau})-2j(u_\tau)-d(u^h-v_h,p)\\
&+b(u_H,u_H,u^h-v_h)-b(u^h,u_H,u^h-v_h)-b(u_H,u^h,u^h-v_h)+j(v_{h\tau})\\
&-a(v_h,u^h-v_h)+d(u^h-v_h,q_h)-d(v_h,p^h-q_h)-G(q_h,p^h-q_h)\\
\le&|a(u-v_h,u^h-v_h)|+|d(u^h-v_h,p-q_h)-d(u-v_h,p^h-q_h)|+|G(q_h,p^h-q_h)|\\
&+|b(u,u,u^h-v_h)-b(u^h,u_H,u^h-v_h)-b(u_H,u^h,u^h-v_h)+b(u_H,u_H,u^h-v_h)|\\
&+|j(v_{h\tau})-2j(u_\tau)+j(2u_\tau-v_{h\tau})|\\
=&I_1+\cdots+I_5.
\end{aligned}\tag{4.13}$$

Now, we begin to estimate I_1 to I_5. In virtue of Young's inequality, I_1 is estimated by

$$I_1 \leq \mu\|u-v_h\|_V\|u^h-v_h\|_V \leq \alpha\|u^h-v_h\|_V^2+\frac{\mu^2}{4\alpha}\|u-v_h\|_V^2, \tag{4.14}$$

where $\alpha>0$ is a sufficiently small constant. Similarly, we estimate I_2 as follows:

$$\begin{aligned}I_2 \leq &\|u-v_h\|_V\|p^h-q_h\|+\|u^h-v_h\|_V\|p-q_h\|\\ \leq&\delta\|p^h-q_h\|^2+\frac{1}{4\delta}\|u-v_h\|_V^2+\alpha\|u^h-v_h\|_V^2+\frac{1}{4\alpha}\|p-q_h\|^2,\end{aligned} \tag{4.15}$$

where $\delta>0$ is a sufficiently small constant. In terms of the definition of the stabilized term $G(\cdot,\cdot):\mathcal{M}\times\mathcal{M}\to\mathbb{R}$, I_3 is estimated by

$$\begin{aligned}I_3=&|G(q_h-p,p^h-q_h)+G(p,p^h-q_h)|\\ \leq&\|p-q_h\|\cdot\|p^h-q_h\|+\|p-\Pi p\|\cdot\|p^h-q_h\|\\ \leq&2\delta\|p^h-q_h\|^2+\frac{1}{4\delta}\|p-q_h\|^2+\frac{1}{4\delta}\|p-\Pi p\|^2.\end{aligned} \tag{4.16}$$

Since $b(u_h,v_h,v_h)=0$, then we rewrite I_4 as follows:

$$\begin{aligned}I_4=&|b(u,u,u^h-v_h)-b(u^h,u_H,u^h-v_h)-b(u_H,u^h,u^h-v_h)+b(u_H,u_H,u^h-v_h)|\\ =&|b(u-u^h,u,u^h-v_h)+b(u^h,u-u^h,u^h-v_h)+b(u^h-u_H,u^h-u_H,u^h-v_h)|\\ =&|b(u-v_h,u,u^h-v_h)+b(v_h-u^h,u,u^h-v_h)+b(u^h,u-v_h,u^h-v_h)\\ &+b(u^h-v_h,v_h-u_H,u^h-v_h)-b(v_h-u_H,u^h-v_h,v_h-u_H)|\\ =&|I_6+\cdots+I_{10}|.\end{aligned} \tag{4.17}$$

According to Young's inequality and (2.3), I_6 is estimated by

$$\begin{aligned}|I_6|=&|b(u-v_h,u,u^h-v_h)|\leq N\|u-v_h\|_V\|u\|_V\|u^h-v_h\|_V\\ \leq&\frac{\mu}{2}\|u-v_h\|_V\|u^h-v_h\|_V\leq\alpha\|u^h-v_h\|_V^2+\frac{\mu}{8\alpha}\|u-v_h\|_V^2.\end{aligned} \tag{4.18}$$

We estimate I_7 and I_8 as follows:

$$|I_7|=|b(v_h-u^h,u,u^h-v_h)|\leq N\|u\|_V\|u^h-v_h\|_V^2\leq\frac{\mu}{2}\|u^h-v_h\|_V^2, \tag{4.19a}$$

$$\begin{aligned}|I_8|=|b(u^h,u-v_h,u^h-v_h)|\leq& N\|u^h\|_V\|u-v_h\|_V\|u^h-v_h\|_V\\ \leq&\alpha\|u^h-v_h\|_V^2+c\|u-v_h\|_V^2,\end{aligned} \tag{4.19b}$$

where $c>0$ depends on N,α and $\|u^h\|_V$. Using (4.3), we estimate I_9 as follows:

$$\begin{aligned}|I_9|=&|b(u^h-v_h,v_h-u_H,u^h-v_h)|\\ \leq&N\|v_h-u_H\|_V\|u^h-v_h\|_V^2\\ \leq&N(\|u-v_h\|_V+\|u-u_H\|_V)\|u^h-v_h\|_V^2\\ \leq&cH\|u^h-v_h\|_V^2,\end{aligned} \tag{4.20}$$

where $c>0$ is independent of h and H.

In terms of $|b(u_h,v_h,w_h)|\le c|\log h|^{1/2}\|u_h\|_V\|v_h\|_V\|w_h\|$ (see [25]), we estimate I_{10} as follows:

$$
\begin{aligned}
|I_{10}|=&b(v_h-u_H,u^h-v_h,v_h-u_H)\\
\le& c|\log h|^{\frac{1}{2}}\|v_h-u_H\|_V\|u^h-v_h\|_V\|v_h-u_H\|\\
\le& c|\log h|^{\frac{1}{2}}(\|u-v_h\|_V+\|u-u_H\|_V)(\|u-v_h\|+\|u-u_H\|)\|u^h-v_h\|_V\\
\le& c|\log h|^{\frac{1}{2}}(h+H)(h^2+H^2)\|u^h-v_h\|_V\\
\le& c|\log h|^{\frac{1}{2}}H^3\|u^h-v_h\|_V.
\end{aligned}
\tag{4.21}
$$

Substituting (4.18)-(4.21) into (4.17) and using Young's inequality, we obtain

$$
I_4\le\frac{3\mu}{4}\|u^h-v_h\|_V^2+(\alpha+cH)\|u^h-v_h\|_V^2+c\|u-v_h\|_V^2+c|\log h|H^6. \tag{4.22}
$$

I_5 is estimated by

$$
I_5=|j(v_{h\tau})-2j(u_\tau)+j(2u_\tau-v_{h\tau})|\le c\|u-v_h\|_{L^2(S)}. \tag{4.23}
$$

Hence, substituting (4.14)-(4.16) and (4.22)-(4.23), for sufficiently small α and H, we obtain

$$
\begin{aligned}
\|u^h-v_h\|_V^2\le& c\|u-v_h\|_V^2+c\|p-q_h\|^2+\|p-\Pi p\|^2\\
&+c\|u-v_h\|_{L^2(S)}+c|\log h|H^6+c\delta\|p^h-q_h\|^2,
\end{aligned}
$$

where $c>0$ is independent of h and H. Thus, there holds

$$
\begin{aligned}
\|u-u^h\|_V\le& c\|u-v_h\|_V+c\|p-q_h\|+\|p-\Pi p\|\\
&+c\|u-v_h\|_{L^2(S)}^{\frac{1}{2}}+c|\log h|^{\frac{1}{2}}H^3+c\delta^{\frac{1}{2}}\|p^h-q_h\|.
\end{aligned}
\tag{4.24}
$$

Next, we estimate $\|p^h-q_h\|$. From Theorem 3.1, we have

$$
\begin{aligned}
\beta_2\|p^h-q_h\|\le& \sup_{(w_h,q_h)\in(\widetilde{\mathcal{V}_h},\mathcal{M}_h)}\frac{\mathcal{B}_h(u^h-v_h,p^h-q_h;w_h,q_h)}{\|w_h\|_V+\|q_h\|}\\
=&\sup_{(w_h,q_h)\in(\widetilde{\mathcal{V}_h},\mathcal{M}_h)}\frac{\mathcal{B}_h(u^h-u,p^h-p;w_h,q_h)+\mathcal{B}_h(u-v_h,p-q_h;w_h,q_h)}{\|w_h\|_V+\|q_h\|}.
\end{aligned}
\tag{4.25}
$$

For all $w_h\in\widetilde{\mathcal{V}_h}$, set $v=u\pm w_h$ in (2.1). Then we have

$$
a(u,w_h)+b(u,u,w_h)-d(w_h,p)=(f,w_h),\quad \forall w_h\in\widetilde{\mathcal{V}_h}. \tag{4.26}
$$

Similarly, we set $v_h = u^h \pm w_h$ in (3.1) and obtain

$$\begin{aligned} &a(u^h,w_h)+b(u_H,u^h,w_h)+b(u^h,u_H,w_h)\\ &\qquad -b(u_H,u_H,w_h)-d(w_h,p^h)=(f,w_h), \quad \forall w_h\in\widetilde{\mathcal{V}_h}. \end{aligned} \tag{4.27}$$

Subtracting (4.26) from (4.27) yields

$$\begin{aligned} &b(u,u,w_h)-b(u_H,u^h,w_h)-b(u^h,u_H,w_h)+b(u_H,u_H,w_h)\\ =&a(u^h-u,w_h)-d(w_h,p^h-p), \quad \forall w_h\in\widetilde{V}_h. \end{aligned} \tag{4.28}$$

Then, in virtue of the definition of $\mathcal{B}_h$, following (4.28), we have

$$\begin{aligned} &\mathcal{B}_h(u^h-u,p^h-p;w_h,q_h)\\ =&a(u^h-u,w_h)-d(w_h,p^h-p)+d(u^h-u,q_h)+G(p^h-p,q_h)\\ =&b(u,u,w_h)+b(u_H,u_H,w_h)-b(u_H,u^h,w_h)-b(u^h,u_H,w_h)-G(p,q_h)\\ =&b(u-u^h,u,w_h)+b(R_hu,u-R_hu,w_h)-b(R_hu-u^h,u-R_hu,w_h)\\ &\quad +b(u^h-u_H,R_hu-u_H,w_h)-b(u_H,u^h-R_hu,w_h)-G(p,q_h)\\ =&I_{11}+\cdots+I_{16}. \end{aligned} \tag{4.29}$$

We estimate I_{11} to I_{16} as follows:

$$|I_{11}|=|b(u-u^h,u,w_h)|\le N\|u-u^h\|_V\|u\|_V\|w_h\|_V\le c\|u-u^h\|_V\|w_h\|_V, \tag{4.30a}$$

$$|I_{12}|=|b(R_hu,u-R_hu,w_h)|\le N\|u-R_hu\|_V\|R_hu\|_V\|w_h\|_V\le ch\|w_h\|_V, \tag{4.30b}$$

$$\begin{aligned} |I_{13}|=&|b(R_hu-u^h,u-R_hu,w_h)|\le N(\|u-R_hu\|_V+\|u-u^h\|_V)\|u-R_hu\|_V\|w_h\|_V\\ &\le ch(h+\|u-u^h\|_V)\|w_h\|_V, \end{aligned} \tag{4.30c}$$

$$\begin{aligned} |I_{14}|=&|b(u^h-u_H,R_hu-u_H,w_h)|\le c|\log h|^{\frac12}\|u^h-u_H\|_V\|w_h\|_V\|R_hu-u_H\|\\ &\le c|\log h|^{\frac12}(\|u^h-u\|_V+\|u-u_H\|_V)(\|u-u_H\|+\|R_hu-u\|)\|w_h\|_V\\ &\le c|\log h|^{\frac12}H^2\|u^h-u\|_V\|w_h\|_V+c|\log h|^{\frac12}H^3\|w_h\|_V, \end{aligned} \tag{4.30d}$$

$$\begin{aligned} |I_{15}|=&|b(u_H,u^h-R_hu,w_h)|\le N\|u_H\|_V\|u^h-R_hu\|_V\|w_h\|_V\\ &\le c(\|u^h-u\|_V+\|u-R_hu\|_V)\|w_h\|_V\\ &\le c\|u^h-u\|_V\|w_h\|_V+ch\|w_h\|_V, \end{aligned} \tag{4.30e}$$

$$|I_{16}|=|G(p,q_h)|\le c\|p-\Pi p\|\|q_h\|\le ch\|q_h\|. \tag{4.30f}$$

Following the above estimates (4.30a)-(4.30f), we obtain

$$\mathcal{B}_h(u^h-u,p^h-p;w_h,q_h)\le c\|u-u^h\|_V\|w_h\|_V+c(h+|\log h|^{\frac12}H^3)\|w_h\|_V++ch\|q_h\|.$$

Thus, from (4.25) and Theorem 3.1, we have

$$\|p^h-q_h\|\le c\|u-u^h\|_V+c(h+|\log h|^{1/2}H^3)+c\|u-v_h\|_V+c\|p-q_h\|. \tag{4.31}$$

Substituting (4.31) into (4.24) and for sufficiently small δ, we get

$$\|u-u^h\|_V \le c(h+|\log h|^{\frac{1}{2}}H^3).$$

Using (4.31) again, we get

$$\|p-p^h\| \le c(h+|\log h|^{\frac{1}{2}}H^3).$$

So, the theorem is proved. □

Remark 4.1. The assumption $|\log h|^{1/2}H^2<1$ holds. In order to obtain the optimal convergence order, in virtue of Theorem 4.1, we choose $h=\mathcal{O}(|\log h|^{1/2}H^3)$. In this case, $|\log h|^{1/2}H^2=\mathcal{O}(h/H)$ with $h\ll H$.

Now, we begin to show the L^2 estimate $\|u-u^h\|$ by the Aubin-Nitsche technique. In order to do that, we need the following regularity assumption about the homogeneous linearized Navier-Stokes equations:

Given $z\in L^2(\Omega)^2$, suppose that the linearized Navier-Stokes equations

$$\begin{cases} \text{find } (w,\pi)\in\mathcal{V}\times\mathcal{M} \ \text{ such that} \\ a(w,v)+b(u_H,v,w)+b(v,u_H,w)-d(v,\pi)=(z,v), \quad \forall v\in\mathcal{V}, \\ d(w,q)=0, \quad \forall q\in\mathcal{M}, \end{cases} \tag{4.32}$$

admits a unique solution $(w,\pi)\in H^2(\Omega)^2\cap\mathcal{V}\times H^1(\Omega)\cap\mathcal{M}$ satisfying

$$\|w\|_2+\|\pi\|_1\le c\|z\|,$$

where $c>0$ is independent of h. Denote $(w_h,\pi_h)\in\widetilde{\mathcal{V}_h}\times\mathcal{M}_h$ the stabilized finite element approximation solution of (4.32). Since u_H is uniformly bounded in $\mathcal{V}$, then there holds

$$\|w-w_h\|_V+\|\pi-\pi_h\|\le ch\|z\|, \tag{4.33}$$

where $c>0$ is independent of h and H. We recall a lemma due to Layton [26,27].

Lemma 4.2. *Suppose $u,u_H\in\mathcal{V}$ and $v\in\mathcal{V}\cap H^2(\Omega)^2$. For every $0<\varepsilon\le 1$, there exists some positive constant $C=C(\varepsilon)$ such that*

$$|b(u-u_H,u-u_H,v)\le C\|u-u_H\|^{2-\varepsilon}\|u-u_H\|_V^{\varepsilon}\|v\|_2, \tag{4.34a}$$

$$|b(u-u_H,u-u_H,v)\le C\|u-u_H\|^{1-\varepsilon}\|u-u_H\|_V^{1+\varepsilon}\|v\|_V. \tag{4.34b}$$

Theorem 4.2. *Under the assumptions in Theorem 4.1, the two-level Newton iteration solution (u^h,p^h) satisfies the following L^2 error estimate:*

$$\|u-u^h\|\le c(h^2+h|\log h|^{\frac{1}{2}}H^3+H^{4-\varepsilon}), \tag{4.35}$$

where $0<\varepsilon\le 1$, $c>0$ is independent of h and H.

Proof. Set $z=v=u-u^h$ in (4.32). Then we get

$$\|u-u^h\|^2=a(u-u^h,w)+b(u_H,u-u^h,w)+b(u-u^h,u_H,w)-d(u-u^h,\pi). \tag{4.36}$$

For the approximation w_h of w, setting $v=u\pm w_h$ in (2.1) and $v_h=u_h\pm w_h$ in (3.1) yields

$$a(u,w_h)+b(u,u,w_h)-d(w_h,p)=(f,w_h), \tag{4.37a}$$

$$a(u^h,w_h)+b(u_H,u^h,w_h)+b(u^h,u_H,w_h)-b(u_H,u_H,w_h)-d(w_h,p^h)=(f,w_h). \tag{4.37b}$$

Subtracting (4.37a) from (4.37b), we obtain

$$\begin{aligned}&a(u-u^h,w_h)-d(w_h,p-p^h)+b(u-u_H,u-u_H,w_h)\\&\quad+b(u_H,u-u^h,w_h)+b(u-u^h,u_H,w_h)=0.\end{aligned}$$

Hence, from (4.36), we have

$$\begin{aligned}\|u-u^h\|^2=&a(u-u^h,w-w_h)+b(u_H,u-u^h,w-w_h)+b(u-u^h,u_H,w-w_h)\\&-b(u-u_H,u-u_H,w_h)-d(w-w_h,p-p^h)-d(u-u^h,\pi)\\=&J_1+\cdots+J_5.\end{aligned} \tag{4.38}$$

About J_1, J_2, J_4, we have

$$|J_1|=|a(u-u^h,w-w_h)|\le\mu\|u-u^h\|_V\|w-w_h\|_V\le ch(h+|\log h|^{\frac{1}{2}}H^3)\|u-u^h\|, \tag{4.39a}$$

$$\begin{aligned}|J_2|=&|b(u_H,u-u^h,w-w_h)+b(u-u^h,u_H,w-w_h)|\\\le&2N\|u_H\|_V\|u-u^h\|_V\|w-w_h\|_V\le ch(h+|\log h|^{\frac{1}{2}}H^3)\|u-u^h\|,\end{aligned} \tag{4.39b}$$

$$|J_4|=|d(w-w_h,p-p^h)|\le\|w-w_h\|_V\|p-p^h\|\le ch(h+|\log h|^{\frac{1}{2}}H^3)\|u-u^h\|. \tag{4.39c}$$

About J_3, using Lemma 4.2, we have

$$\begin{aligned}|J_3|=&|b(u-u_H,u-u_H,w_h)|\\\le&|b(u-u_H,u-u_H,w_h-w)|+|b(u-u_H,u-u_H,w)|\\\le&C\|u-u_H\|^{1-\varepsilon}\|u-u_H\|_V^{1+\varepsilon}\|w-w_h\|_V+C\|u-u_H\|^{2-\varepsilon}\|u-u_H\|_V^{\varepsilon}\|w\|_2\\\le&chH^{3-\varepsilon}\|u-u^h\|+cH^{4-\varepsilon}\|u-u^h\|\\\le&cH^{4-\varepsilon}\|u-u^h\|.\end{aligned} \tag{4.40}$$

About J_5, we note that $\Pi\pi$ is the piecewise constant, then

$$d(u-u_h,\Pi\pi)=-\int_\Omega(u-u_h)\cdot\nabla\Pi\pi\mathrm{d}x+\int_S(u-u_h)\cdot n\Pi\pi\mathrm{d}s\equiv 0.$$

Thus, we obtain

$$\begin{aligned}|J_5|=&|d(u-u_h,\pi)|=|d(u-u_h,\pi-\Pi\pi)|\\\le&\|u-u^h\|_V\|\pi-\Pi\pi\|\le ch(h+|\log h|^{\frac{1}{2}}H^3)\|u-u^h\|.\end{aligned} \tag{4.41}$$

Substituting (4.39a)-(4.41) into (4.38), we obtain

$$\|u-u^h\| \le c(h^2+h|\log h|^{\frac{1}{2}}H^3+H^{4-\varepsilon}).$$

The proof is completed. $\square$

Remark 4.2. The L^2 error estimate is suboptimal even if we choose

$$h=\mathcal{O}(|\log h|^{1/2}H^3).$$

Hence, $H^3=\mathcal{O}(h|\log h|^{-1/2})$. Then we have $H^{4-\varepsilon}=\mathcal{O}(H^{1-\varepsilon}h|\log h|^{-1/2})$. Note that $|\log h|^{-1/2}<1$ when h is sufficiently small. Thus, the estimate (4.35) becomes

$$\|u-u^h\| \le c(h^2+H^{1-\varepsilon}h). \tag{4.42}$$

5 Numerical results

In this section, we will give the numerical results to support the theoretical results derived in Theorems 4.1 and 4.2. However, the discrete problems (4.1)-(4.2) on the coarse mesh and on the fine mesh are the variational inequality problems. Then, we must construct the numerical iteration schemes to solve these variational inequality problems. Here, we use the Uzawa iteration method, which has been used to solve the numerical solution of the Stokes type variational inequality problem in [10,11,28].

However, we only give the Uzawa iteration method for the variational inequality problem (2.1). The similar method can be used to solve the two-level stabilized schemes (4.1)-(4.2). The variational inequality problem (2.1) is equivalent to the following variational equation:

$$\begin{cases} a(u,v)+b(u,u,v)-d(v,p)+\int_S \lambda g v_\tau \mathrm{d}s=(f,v), & \forall v\in\mathcal{V},\\ d(u,q)=0, & \forall q\in\mathcal{M},\\ \lambda u_\tau=|u_\tau|, & \text{a.e. on } S, \end{cases}$$

where $\lambda\in\Lambda=\{\gamma\in L^2(S):\ |\gamma(x)|\le 1 \text{ a.e. on } S\}$. In this case, we can solve the variational inequality problem (2.1) by the following Uzawa iteration scheme:

$$\lambda^0\in\Lambda \ \text{ is given,} \tag{5.1}$$

then as λ^n is known, we compute (u^n,p^n) and λ^{n+1} by

$$\begin{cases} a(u^n,v)+b(u^n,u^n,v)-d(v,p^n)=(f,v)-\int_S \lambda^n g v_\tau \mathrm{d}s, & \forall v\in\mathcal{V},\\ d(u^n,q)=0, & \forall q\in\mathcal{M}, \end{cases} \tag{5.2}$$

and

$$\lambda^{n+1}=P_\Lambda(\lambda^n+\rho g u_\tau^n), \tag{5.3}$$

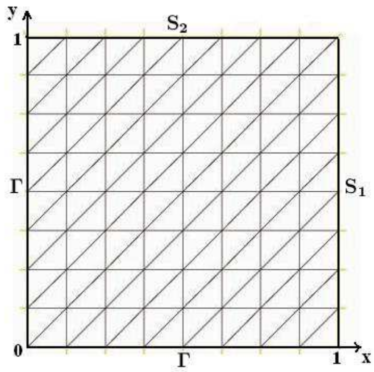

Figure 1: The domain Ω.

where $\rho > 0$ is a parameter, P_Λ from $L^2(S)$ to Λ is the projection operator defined by

$$P_\Lambda(\gamma) = \sup\{-1, \inf(1,\gamma)\}, \quad \forall \gamma \in L^2(S).$$

Consider the problem (2.1) in the fixed square domain $(0,1)\times(0,1)$ (see Fig. 1). Choose the appropriate f such that the exact solution (u,p) is given by

$$\begin{aligned} &u(x,y) = (u_1(x,y), u_2(x,y)), && p(x,y) = (2x-1)(2y-1), \\ &u_1(x,y) = -x^2 y(x-1)(3y-2), && u_2(x,y) = xy^2(y-1)(3x-2). \end{aligned}$$

For all $p_h, q_h \in M_h$, the stabilized term $G(p_h,q_h)$ in the finite element approximation formulation can be computed by the following local Gauss integration method in [18]:

$$G(p_h,q_h) = p_i^T (M_k - M_1) q_j = p_i^T M_k q_j - p_i^T M_1 q_j,$$

where

$$\begin{aligned} &p_i^T = [p_0, p_1, \cdots, p_{N-1}]^T, \qquad q_j = [q_0, q_1, \cdots, q_{N-1}], \\ &M_{ij} = (\phi_i, \phi_j), \quad p_h = \sum_{i=0}^{N-1} p_i \phi_i, \quad p_i = p_h(x_i), \quad i,j = 0,1,\cdots, N-1, \end{aligned}$$

where ϕ_i is the base function in Ω with respect to the pressure such that its value is one at the node x_i and is zero at other nodes. M_k, $k \geq 2$ and M_1 are pressure mass matrix computed by using k-order and 1-order Gauss integration in each direction, respectively. p_i, q_i are the value of p_h, q_h at node x_i. p_i^T is the transpose of the matrix p_i.

Let the iteration initial value $\lambda^0 = 1$ and the parameter $\rho = \mu = 0.1$ in (5.3). We pick nine coarse mesh values $H = 1/2, 1/3, \cdots, 1/10$. In terms of Theorem 4.1, we choose the fine mesh $h = \mathcal{O}(|\log h|^{1/2} H^3)$ such that the error derived in Theorem 4.1 is of the optimal convergence order:

$$\|u - u^h\|_V + \|p - p^h\| \leq ch. \tag{5.4}$$

Table 1: Values of h.

$1/H$	2	3	4	5	6	7	8	9	10
$1/h$	8	16	32	61	101	153	221	305	408

Table 2: Relative errors and their convergence orders.

$1/H$	$1/h$	$\frac{\lVert u-u^h\rVert_V}{\lVert u\rVert_V}$	Order	$\frac{\lVert u-u^h\rVert}{\lVert u\rVert}$	Order	$\frac{\lVert p-p^h\rVert}{\lVert p\rVert}$	Order	CPU
2	8	0.313701	/	0.0848167	/	0.0557971	/	0.469
3	16	0.151545	1.0496	0.0213492	1.9902	0.0207313	1.4284	0.235
4	32	0.073768	1.0387	0.0049929	2.0962	0.0073802	1.4901	0.89
5	61	0.038159	1.0217	0.0015329	1.8304	0.0028017	1.5013	3.203
6	101	0.022894	1.0131	0.0005971	1.8695	0.0013129	1.5032	8.938
7	153	0.015061	1.0084	0.0002908	1.7322	0.0007034	1.5024	22.047
8	221	0.010404	1.0059	0.0001528	1.7488	0.0004057	1.5016	44.188
9	305	0.007527	1.0041	0.0000891	1.6756	0.0002497	1.5011	89.782
10	408	0.005623	1.0031	0.0000547	1.6739	0.0001614	1.4999	159.672

From Remark 4.2, the L^2 error estimate $\|u-u^h\|$ is suboptimal. Table 1 displays the values of h with corresponding to the H.

Tables 2 displays the relative H^1 and L^2 errors of the velocity and the relative L^2 error of the pressure and their convergence orders and CPU time, from which we observe the predicted optimal convergence orders of $\|u-u^h\|_V$ and $\|p-p^h\|$. However, the convergence order of $\|u-u^h\|$ becomes smaller and smaller as H and h decrease, which support the estimate (4.42). The CPU time implies that the two-level Newton iteration method is an efficient and high-performance algorithm to solve the variational inequality problem (2.1).

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