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A Galerkin Splitting Symplectic Method for the Two Dimensional Nonlinear Schrödinger Equation

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Abstract. In this paper, we propose a Galerkin splitting symplectic (GSS) method for solving the 2D nonlinear Schrödinger equation based on the weak formulation of the equation. First, the model equation is discretized by the Galerkin method in spatial direction by a selected finite element method and the semi-discrete system is rewritten as a finite-dimensional canonical Hamiltonian system. Then the resulted Hamiltonian system is split into a linear Hamiltonian subsystem and a nonlinear subsystem. The linear Hamiltonian subsystem is solved by the implicit midpoint method and the nonlinear subsystem is integrated exactly. By the Strang splitting method, we obtain a fully implicit scheme for the 2D nonlinear Schrödinger equation (NLS), which is symmetric and of order 2 in time. Furthermore, we apply the FFT technique to improve computation efficiency of the new scheme. It is proven that our scheme preserves the mass conservation and the symplectic conservation. Comprehensive numerical experiments are carried out to illustrate the accuracy of the scheme as well as its conservative properties.

AMS subject classifications: 65M06, 65M20, 65M70

Key words: Hamiltonian PDEs, nonlinear Schrödinger equation in two dimensions, symplectic integrator, Galerkin method, the Strang splitting method.

1 Introduction

Hamiltonian system is widely used in many important fields such as quantum field theory, electromagnetism, oceanography, meteorology, optics, astronomy and mechanics and so on. Feng [1] proposed the symplectic algorithm which was robust, efficient

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and very accurate in preserving the long-time behavior of solutions of Hamiltonian systems. Subsequently, extensive numerical studies about symplectic methods have been carried out for Hamiltonian ordinary differential equations (ODEs) in the literature, see, e.g., [2–6] for details. Naturally, we hope that the frame work of symplectic method is extended for solving Hamiltonian partial differential equations (PDEs). Yet, we note there is a major difficulty because the underlying phase space is extended from finite dimensions to infinite dimensions. The method of line is a popular method to treat Hamiltonian PDEs. Its main idea is to first discretize PDEs in spatial direction resulting in a large system of Hamiltonian ODEs, then the resulting ODEs are approximated by a symplectic algorithm [7, 8]. To our knowledge, most existing symplectic methods for Hamiltonian PDEs, are mainly based on the finite difference method [9, 10], the wavelet collocation method [11, 12], the Fourier pseudospectral method [13–15]. In contrast, the study of symplectic methods constructed from finite element method (FEM) type approaches are still in its infancy, although some cases for FEM methods have been examined in [16–19]. Thus, we aim to develop a new symplectic scheme base on FEM method. Furthermore, we note that most of the symplectic schemes are completely implicit, which will greatly suppress the efficiency. To overcome this disadvantage, we introduce the splitting method to save CPU time and computer memory. Thus, improving computation efficiency by the splitting method and the FFT technique is the second aim in this paper.

However, in spite of quite a number of contributions dealing with 2D Hamiltonian equations, there are a few symplectic schemes for 2D Hamiltonian equations in the literature [20,21]. Here, we take the nonlinear Schrödinger (NLS) equation in two dimensions as an example to construct symplectic scheme. The nonlinear Schrödinger equation has wide applications in many fields of physics, such as the evolution of waves in water and plasma, optical pulse and so on [22–24]. A large number of mathematical and numerical results have been achieved for the NLS equation in the literature. Along the mathematical front, for the derivation, well-posedness and dynamical properties of the NLS equation, one see, e.g., [25,26] for details. As for the numerical methods for the NLS equations, the time-splitting pseudo-spectral method [27–29], finite difference method [30,31] and finite element method [32–35] and so on, have been well-developed.

In this paper, we aim to construct a novel numerical method to preserve the discrete symplectic conservation law for the 2D-NLS, which discretizes the equation by the Galerkin method in space and the semi-discrete system is rewritten as a finite-dimensional canonical Hamiltonian system. Then the obtained semi-discrete Hamiltonian system is split into a linear Hamiltonian subsystem and a nonlinear subsystem. The linear subsystem is approximated by the implicit midpoint method in time and the nonlinear subsystem is solved exactly due to their mass conservation law. A fully discrete scheme is derived by the Strang splitting method. We rigorously prove that the new scheme hold the mass conservation and the symplectic conservation in the discrete version. Furthermore, the Fast Fourier transform (FFT) algorithm and the matrix diagonalization method are used to solve the discrete system to enhance the computational