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Some Invariant Solutions of Two-Dimensional Elastodynamics in Linear Homogeneous Isotropic Materials

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Abstract. Invariant solutions of two-dimensional elastodynamics in linear homogeneous isotropic materials are considered via the group theoretical method. The second order partial differential equations of elastodynamics are reduced to ordinary differential equations under the infinitesimal operators. Three invariant solutions are constructed. Their graphical figures are presented and physical meanings are elucidated in some cases.

AMS subject classifications: 74B05, 35Q74, 22E70

Key words: Elastodynamics, group theoretical method, invariant solution.

1 Introduction

Elastodynamics is one of the oldest topics in the theory of elasticity. It began almost 200 years ago when Navier announced the general equations of equilibrium and motion of an isotropic elastic body in 1821. However, till today, known exact solutions of elastodynamics are still very limited. In [1, 2], Kausel and Kachanov collected some exact solutions for classical and canonical problems in elastodynamics. The group theoretical method is a very powerful and versatile tool to find invariant solutions of differential equations, especially partial differential equations. It provides two basic ways: group transformation of known solutions and construction of invariant solutions. Chand [3] studied invariant solutions of one-dimensional wave propagation in dissipative materials. For a one-dimensional system of wave propagation equations in linear, viscoelastic and viscoplastic material, Ames and Suliciu constructed its invariant solutions in [4] and

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Ames obtained its group properties and conservation laws in [5]. Bokhari [6] studied the invariant solutions of a nonlinear wave equation. It is clear that the group theoretical method is also very effective in solid mechanics. The above authors studied (1+1)-dimensional problems in the space $R^{1+1}(x,t)$ by the group theoretical method. In this paper, we consider invariant solutions of (2+1)-dimensional elastodynamics in linear homogeneous isotropic materials in the space $R^{2+1}(x,y,t)$.

2 The governing equations for two-dimensional elastodynamics

Two-dimensional elastic body occupies the domain as a plane Ω. Displacement boundary is denoted by $\partial_u\Omega$. ρ is the mass density of the elastic body. λ, μ are Lame's coefficients. In the Cartesian coordinate system (x_1,x_2), displacement vector is $\mathbf{u}=(u_1,u_2)$. The theory of elastodynamics specializes in the case when all fields are time-dependent, t is time variable. In homogeneous isotropic media, the linear theory of two-dimensional elastodynamics without body force is the following equations in term of displacement $\mathbf{u}=\mathbf{u}(x_1,x_2,t)$

$$\begin{cases} (\lambda+\mu)\dfrac{\partial}{\partial x_1}\Big(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\Big)+\mu\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\Big)u_1=\rho\dfrac{\partial^2 u_1}{\partial t^2}, \\ (\lambda+\mu)\dfrac{\partial}{\partial x_2}\Big(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\Big)+\mu\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\Big)u_2=\rho\dfrac{\partial^2 u_2}{\partial t^2}. \end{cases} \tag{2.1}$$

In order to facilitate the research invariant solutions of Eq. (2.1), we now non-dimensionalize Eq. (2.1) with the characteristic length l,

$$x_i^*=\frac{x_i}{l}, \tag{2.2}$$

where i, $j=1,2$. Substituting the dimensionless quantities into Eq. (2.1) and removing the coordinate's asterisks *, Eq. (2.1) are transformed to

$$\begin{cases} \dfrac{\partial}{\partial x_1}\Big(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\Big)+\beta\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\Big)u_1=\dfrac{\partial^2 u_1}{\partial \tau^2}, \\ \dfrac{\partial}{\partial x_2}\Big(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\Big)+\beta\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\Big)u_2=\dfrac{\partial^2 u_2}{\partial \tau^2}, \end{cases} \tag{2.3}$$

where

$$\begin{cases} \tau=t(\lambda+\mu)^{\frac{1}{2}}\rho^{-\frac{1}{2}}/l, \\ \beta=\mu/(\lambda+\mu)=1-2\nu. \end{cases} \tag{2.4}$$

In accordance with the strain energy is positive definite, we deduce the Poisson's ratio ν in the range from -1 to $1/2$. Thus the range of β is (0,3). Generally ν should be