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A Spectral Method for Second Order Volterra Integro-Differential Equation with Pantograph Delay

Weishan Zheng and Yanping Chen*

School of Mathematics sciences, South China Normal University, Guangzhou 510631, Guangdong, China

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Abstract. In this paper, a Legendre-collocation spectral method is developed for the second order Volterra integro-differential equation with pantograph delay. We provide a rigorous error analysis for the proposed method. The spectral rate of convergence for the proposed method is established in both L^2-norm and L^∞-norm.

AMS subject classifications: 65R20, 34K28

Key words: Legendre-spectral method, second order Volterra integro-differential equation, pantograph delay, error analysis.

1 Introduction

The paper is concerned with the second order Volterra integro-differential equation with pantograph delay:

$$u^{(2)}(x)=\sum_{j=0}^{1}a_j(x)u^{(j)}(q_jx)+\sum_{j=0}^{1}b_j(x)u^{(j)}(x)+\sum_{j=0}^{1}\int_0^x k_j(x,s)u^{(j)}(s)ds+g(x), \qquad x\in[0,T], \tag{1.1}$$

with

$$u(0)=u_0, \qquad u^{(1)}(0)=u_1. \tag{1.2}$$

Here, we denote $u^{(j)}(x)=(d^j/dx^j)u(x)$, $j=0,1,2$. q_j is a given constant and $0<q_j<1$. $a_j(x)$, $b_j(x)$, $g(x)$ are smooth functions on [0,T]. $k_j(x,s)$ is also a smooth function on $D(D:=$

*Corresponding author.
URL: `http://math.xtu.edu.cn/myphp/math/ypchen/index.htm`
Email: weishanzheng@yeah.net (W. Zheng), yanpingchen@scnu.edu.cn (Y. Chen)

$\{(x,s):0\le s\le x\le T\}$) and $u^{(j)}(q_jx)$ is pantograph delay, $j=0,1$. $u(x)$ is an unknown function.

Since Volterra integro-differential equations with pantograph delay arise widely in the mathematical model of physical and biological phenomena, many researchers have developed theoretical and numerical analysis for the related types of equations. We refer the reader to [3, 6, 10] for a survey of early results on Volterra integro-differential equations. More recently, polynomial spline collocation methods were investigated in [16, 19] and homotopy analysis method was used to solve system of Volterra integral equations (see, e.g., [14]). In [20, 21], the authors used spectral collocation methods studying convergence analysis about Volterra integro-differential equations. For pantograph delay differential equations, in [2, 12, 23], the authors researched on these kinds of functions. In [1], spectral method was used to solve $y'(x)=a(x)y(qx)$, but it only analysed the numerical error in the infinity norm.

So far, very few work have touched the spectral approximation to second order Volterra integro-differential equations with pantograph delay. In practice, spectral method has excellent convergence property of exponential convergence rate. In this paper, we will provide a Legendre-collocation spectral method for the second order Volterra integro-differential equation with pantograph delay and analyse the numerical error decay exponentially in both L^2 and L^∞ space norms.

For ease of analysis, we will describe the spectral method on the standard interval $[-1,1]$. Hence, we employ the transformation

$$x=\frac{T}{2}(1+t),\qquad t=\frac{2x}{T}-1.$$

Then the above problem (1.1)-(1.2) becomes

$$\begin{aligned} y^{(2)}(t)=&\sum_{j=0}^{1}A_j(t)y^{(j)}(q_jt+q_j-1)+\sum_{j=0}^{1}B_j(t)y^{(j)}(t)\\ &+\sum_{j=0}^{1}\int_0^{\frac{T}{2}(1+t)}\hat{k}_j(t,s)u^{(j)}(s)ds+G(t),\qquad t\in[-1,1], \end{aligned}\tag{1.3}$$

with

$$y(-1)=u_0,\qquad y^{(1)}(-1)=\Big(\frac{T}{2}\Big)u_1,\tag{1.4}$$

where

$$\begin{aligned} &y(t)=u\Big(\frac{T}{2}(1+t)\Big), &&G(t)=\Big(\frac{T}{2}\Big)^2g\Big(\frac{T}{2}(1+t)\Big),\\ &A_0(t)=\Big(\frac{T}{2}\Big)^2a_0\Big(\frac{T}{2}(1+t)\Big), &&A_1(t)=\Big(\frac{T}{2}\Big)a_1\Big(\frac{T}{2}(1+t)\Big), \end{aligned}$$

$$B_0(t)=\Big(\frac{T}{2}\Big)^2 b_0\Big(\frac{T}{2}(1+t)\Big), \qquad B_1(t)=\Big(\frac{T}{2}\Big) b_1\Big(\frac{T}{2}(1+t)\Big),$$
$$\hat{k}_0(t,s)=\Big(\frac{T}{2}\Big)^2 k_0\Big(\frac{T}{2}(1+t),s\Big), \qquad \hat{k}_1(t,s)=\Big(\frac{T}{2}\Big)^2 k_1\Big(\frac{T}{2}(1+t),s\Big).$$

Furthermore, to transfer the integral interval $[0,T(1+t)/2]$ to the interval $[-1,t]$, we make a linear transformation: $s=T(1+\tau)/2$, $\tau\in[-1,t]$. Then, Eq. (1.3) becomes

$$\begin{aligned} y^{(2)}(t)=&\sum_{j=0}^{1} A_j(t)y^{(j)}(q_jt+q_j-1)+\sum_{j=0}^{1} B_j(t)y^{(j)}(t)\\ &+\sum_{j=0}^{1}\int_{-1}^{t} K_j(t,\tau)y^{(j)}(\tau)d\tau+G(t), \qquad t\in[-1,1], \end{aligned} \tag{1.5}$$

where

$$K_0(t,\tau)=\Big(\frac{T}{2}\Big)\hat{k}_0(t,s)=\Big(\frac{T}{2}\Big)^3 k_0\Big(\frac{T}{2}(1+t),\frac{T}{2}(1+\tau)\Big),$$
$$K_1(t,\tau)=\hat{k}_1(t,s)=\Big(\frac{T}{2}\Big)^2 k_1\Big(\frac{T}{2}(1+t),\frac{T}{2}(1+\tau)\Big).$$

The main purpose of this work is to provide a spectral collocation method for the second order Volterra integro-differential equation with pantograph delay. We will provide a rigorous error analysis which theoretically justifies the spectral rate of convergence. This paper is organized as follows. In Section 2, we introduce the spectral approach for the second order Volterra integro-differential equation with pantograph delay. Some useful lemmas are provided in Section 3, which are important for the convergence analysis. In Section 4, we provide the convergence analysis in both L^2 and L^∞ spaces.

Throughout the paper C will denote a generic positive constant that is independent of N, but depends on T and the given data.

2 The spectral method

As demonstrated in the last section, we can assume that the solution domain is $[-1,1]$. The second order Volterra integro-differential equation with pantograph delay in one-dimension are of the form (2.1), namely,

$$\begin{aligned} y^{(2)}(t)=&\sum_{j=0}^{1} A_j(t)y^{(j)}(q_jt+q_j-1)+\sum_{j=0}^{1} B_j(t)y^{(j)}(t)\\ &+\sum_{j=0}^{1}\int_{-1}^{t} K_j(t,s)y^{(j)}(s)ds+G(t), \qquad t\in[-1,1], \end{aligned} \tag{2.1}$$

with

$$y(-1)=y_{-1}, \qquad y^{(1)}(-1)=y'_{-1}, \tag{2.2}$$

where $y_{-1}=u_0$, $y'_{-1}=(T/2)u_1$.

For a given positive integer N, we denote the collocation points by $\{t_i\}_{i=0}^N$, which is the set of the $(N+1)$ Legendre Gauss points, and by $\{\omega_i\}_{i=0}^N$ the corresponding weights. Let $\mathcal{P}_N$ denotes the space of all polynomials of degree not exceeding N. For any $v\in \mathcal{C}[-1,1]$ (see, e.g., [9,11,17]), we can define the Lagrange interpolating polynomial $I_N v\in \mathcal{P}_N$, satisfying

$$I_N v(t_i)=v(t_i), \qquad 0\le i\le N.$$

The Lagrange interpolating polynomial can be written in the form

$$I_N v(t)=\sum_{i=0}^N v(t_i)F_i(t),$$

where $\{F_i(t)\}_{i=0}^N$ is the Lagrange interpolation basis functions associated with the Legendre collocation points $\{t_i\}_{i=0}^N$.

In order that the spectral collocation method will be carried out naturally, integrating Eq. (2.1) and using (2.2), we get

$$\begin{aligned} y^{(1)}(t)=y'_{-1}+&\sum_{j=0}^1\int_{-1}^t A_j(s)y^{(j)}(q_js+q_j-1)ds \\ &+\sum_{j=0}^1\int_{-1}^t B_j(s)y^{(j)}(s)ds+\int_{-1}^t v(s)ds, \end{aligned} \tag{2.3a}$$

$$y(t)=y_{-1}+\int_{-1}^t y^{(1)}(s)ds, \tag{2.3b}$$

$$v(t)=\sum_{j=0}^1\int_{-1}^t K_j(t,s)y^{(j)}(s)ds+G(t). \tag{2.3c}$$

Firstly, Eqs. (2.3a)-(2.3c) hold at the collocation points $\{t_i\}_{i=0}^N$ on $[-1,1]$, namely,

$$\begin{aligned} y^{(1)}(t_i)=y'_{-1}+&\sum_{j=0}^1\int_{-1}^{t_i} A_j(s)y^{(j)}(q_js+q_j-1)ds \\ &+\sum_{j=0}^1\int_{-1}^{t_i} B_j(s)y^{(j)}(s)ds+\int_{-1}^{t_i} v(s)ds, \end{aligned} \tag{2.4a}$$

$$y(t_i)=y_{-1}+\int_{-1}^{t_i} y^{(1)}(s)ds, \tag{2.4b}$$

$$v(t_i)=\sum_{j=0}^1\int_{-1}^{t_i} K_j(t_i,s)y^{(j)}(s)ds+G(t_i), \tag{2.4c}$$

for $0\le i\le N$. In order to obtain high order accuracy of the approximated solution, we have to compute the integral term. In particular, for small value of t_i, there is little information available for $y(s)$ and $y^{(1)}(s)$. To overcome this difficulty, we transfer the integral interval $[-1,t_i]$ to $[-1,1]$ by using the following variable change:

$$s=\frac{1+t_i}{2}\theta+\frac{t_i-1}{2}\triangleq s(t_i,\theta),\qquad \theta\in[-1,1]. \tag{2.5}$$

Then the above equations become:

$$\begin{aligned} y^{(1)}(t_i)=&\,y'_{-1}+\frac{t_i+1}{2}\sum_{j=0}^{1}\int_{-1}^{1}A_j(s(t_i,\theta))y^{(j)}(q_js(t_i,\theta)+q_j-1)d\theta\\ &+\frac{t_i+1}{2}\sum_{j=0}^{1}\int_{-1}^{1}B_j(s(t_i,\theta))y^{(j)}(s(t_i,\theta))d\theta+\frac{t_i+1}{2}\int_{-1}^{1}v(s(t_i,\theta))d\theta, \end{aligned} \tag{2.6a}$$

$$y(t_i)=y_{-1}+\frac{t_i+1}{2}\int_{-1}^{1}y^{(1)}(s(t_i,\theta))d\theta, \tag{2.6b}$$

$$v(t_i)=\frac{t_i+1}{2}\sum_{j=0}^{1}\int_{-1}^{1}K_j(t_i,s(t_i,\theta))y^{(j)}(s(t_i,\theta))d\theta+G(t_i). \tag{2.6c}$$

Next, using an $(N+1)$-point Gauss quadrature formula relative to the Legendre weights $\{\omega_i\}_{i=0}^N$ to approximate the integration term, we get

$$\begin{aligned} y_i^{(1)}=&\,y'_{-1}+\frac{t_i+1}{2}\sum_{j=0}^{1}\Big(\sum_{k=0}^{N}A_j(s(t_i,\theta_k))y^{(j)}(q_js(t_i,\theta_k)+q_j-1)w_k\Big)\\ &+\frac{t_i+1}{2}\sum_{j=0}^{1}\Big(\sum_{k=0}^{N}B_j(s(t_i,\theta_k))y^{(j)}(s(t_i,\theta_k))w_k\Big)+\frac{t_i+1}{2}\sum_{k=0}^{N}v(s(t_i,\theta_k))w_k, \end{aligned} \tag{2.7a}$$

$$y_i=y_{-1}+\frac{t_i+1}{2}\sum_{k=0}^{N}y^{(1)}(s(t_i,\theta_k))w_k, \tag{2.7b}$$

$$v_i=\frac{t_i+1}{2}\sum_{j=0}^{1}\Big(\sum_{k=0}^{N}K_j(t_i,s(t_i,\theta_k))y^{(j)}(s(t_i,\theta_k))w_k\Big)+G(t_i), \tag{2.7c}$$

where $y_i^{(1)}\approx y^{(1)}(t_i)$, $y_i\approx y(t_i)$ and $v_i\approx v(t_i)$, $0\le i\le N$. The set $\{\theta_k\}_{k=0}^N$ coincides with the collocation points $\{t_i\}_{i=0}^N$.

We expand $y^{(1)}$, y and v using Lagrange interpolation polynomials, i.e.,

$$y^{(1)}(s)\approx\sum_{p=0}^{N}y_p^{(1)}F_p(s),\qquad y(s)\approx\sum_{p=0}^{N}y_pF_p(s),\qquad v(s)\approx\sum_{p=0}^{N}v_pF_p(s).$$

The Legendre collocation method is to seek $\{y_i^{(1)}\}_{i=0}^N,\{y_i\}_{i=0}^N$ and $\{v_i\}_{i=0}^N$ such that the following collocation equations hold

$$\begin{aligned} y_i^{(1)} =& y'_{-1} + \frac{t_i+1}{2}\sum_{j=0}^{1}\Big(\sum_{p=0}^{N} y_p^{(j)} \sum_{k=0}^{N} A_j(s(t_i,\theta_k))F_p(q_j s(t_i,\theta_k)+q_j-1)w_k\Big) \\ &+ \frac{t_i+1}{2}\sum_{j=0}^{1}\Big(\sum_{p=0}^{N} y_p^{(j)} \sum_{k=0}^{N} B_j(s(t_i,\theta_k))F_p(s(t_i,\theta_k))w_k\Big) \\ &+ \frac{t_i+1}{2}\sum_{p=0}^{N} v_p \sum_{k=0}^{N} F_p(s(t_i,\theta_k))w_k, \end{aligned} \tag{2.8a}$$

$$y_i = y_{-1} + \frac{t_i+1}{2}\sum_{p=0}^{N} y_p^{(1)} \sum_{k=0}^{N} F_p(s(t_i,\theta_k))w_k, \tag{2.8b}$$

$$v_i = \frac{t_i+1}{2}\sum_{j=0}^{1}\Big(\sum_{p=0}^{N} y_p^{(j)} \sum_{k=0}^{N} K_j(t_i,s(t_i,\theta_k))F_p(s(t_i,\theta_k))w_k\Big) + G(t_i). \tag{2.8c}$$

Remark 2.1. Generally, the analysis of Volterra integro-differential equations with pantograph delay can be based on the integral equations that are equivalent to the original initial-value problem. This reformulation will not affect the regularity properties of solutions and the accuracy of numerical solutions (see, e.g., [5, 7, 13]). Consequently, it is reasonable that we consider the equivalent reformulation system (2.3a)-(2.3c). This transformation does not influence the stability of Legendre-collocation spectral approximation scheme (see, e.g., [9, 19]).

3 Some useful lemmas

In this section, we will provide some elementary lemmas, which are important for the derivation of the main results in the subsequent section.

Lemma 3.1. (*see* [9]) *Assume that an* $(N+1)$*-point Gauss quadrature formula relative to the Legendre weight is used to integrate the product* $y\phi$*, where* $y\in H^m(I)$ *with* $I:=(-1,1)$ *for some* $m\geq 1$ *and* $\phi\in\mathcal{P}_N$*. Then there exists a constant C independent of N such that*

$$\Big|\int_{-1}^{1} y(t)\phi(t)dt - (y,\phi)_N\Big| \leq CN^{-m}|y|_{H^{m,N}(I)}\|\phi\|_{L^2(I)}, \tag{3.1}$$

where

$$|y|_{H^{m,N}(I)} = \Big(\sum_{k=\min(m,N+1)}^{m} \|y^{(k)}\|^2_{L^2(I)}\Big)^{\frac{1}{2}}, \tag{3.2a}$$

$$(y,\phi)_N = \sum_{j=0}^{N} y(t_j)\phi(t_j)\omega_j. \tag{3.2b}$$

Lemma 3.2. (*see* [9]) *Assume that* $y\in H^m(I)$ *with* $I:=(-1,1)$ *and denote* I_Ny *its interpolation polynomial associated with the* $(N+1)$*-point Gauss points* $\{t_i\}_{i=0}^N$*, namely,*

$$I_Ny=\sum_{i=0}^{N}y(t_i)F_i(t). \tag{3.3}$$

Then the following estimates hold

$$\|y-I_Ny\|_{L^2(I)}\leq CN^{-m}|y|_{H^{m,N}(I)}, \tag{3.4a}$$

$$\|y-I_Ny\|_{H^l(I)}\leq CN^{2l-1/2-m}|y|_{H^{m,N}(I)}, \qquad 1\leq l\leq m. \tag{3.4b}$$

Lemma 3.3. (*see* [8]) *For every bounded function* $v(t)$*, there exists a constant* C *independent of* v *such that*

$$\sup_N\Big\|\sum_{j=0}^{N}v(t_j)F_j(t)\Big\|_{L^2(I)}\leq C\|v\|_{L^\infty(I)}. \tag{3.5}$$

Lemma 3.4 (Gronwall inequality)**.** *If a non-negative integrable function* $E(t)$ *satisfies*

$$E(t)\leq C_1\int_{-1}^{t}E(s)ds+G(t), \qquad -1<t\leq 1, \tag{3.6}$$

where $G(t)$ *is an integrable function, then*

$$\|E\|_{L^p(I)}\leq C\|G\|_{L^p(I)}, \qquad p\geq 1. \tag{3.7}$$

From (3.6)*, if we have*

$$E(t)\leq C_1\int_{-1}^{t}E(qs+q-1)ds+C_2\int_{-1}^{t}E(s)ds+G(t), \qquad -1<t\leq 1, \tag{3.8}$$

where q *is a constant and* $0<q<1$*. Then we also get*

$$\|E\|_{L^p(I)}\leq C\|G\|_{L^p(I)}, \qquad p\geq 1. \tag{3.9}$$

Proof. Using the following variable change

$$x=qs+q-1, \qquad s=\frac{x}{q}+\frac{1-q}{q}.$$

Note that $qt+q-1=q(t+1)-1<t$ for $0<q<1$ and $t\in(0,1]$, we have

$$\int_{-1}^{t}E(qs+q-1)ds=\frac{1}{q}\int_{-1}^{qt+q-1}E(x)dx<\frac{1}{q}\int_{-1}^{t}E(x)dx=C\int_{-1}^{t}E(s)ds. \tag{3.10}$$

This together with (3.8), we get

$$\begin{aligned} E(t) \le & C_1 \int_{-1}^{t} E(qs+q-1)ds + C_2 \int_{-1}^{t} E(s)ds + G(t) \\ \le & C \int_{-1}^{t} E(s)ds + G(t). \end{aligned} \tag{3.11}$$

That leads to the result of (3.9). □

Lemma 3.5. (*see* [15]) *Assume that $F_j(t)$ is the j-th Lagrange interpolation polynomial associated with the Legendre-Gauss points. Then*

$$\max_{t\in(-1,1)} \sum_{j=0}^{N} |F_j(t)| = 1 + \frac{2^{\frac{3}{2}}}{\sqrt{\pi}} N^{\frac{1}{2}} + B_0 + \mathcal{O}(N^{-\frac{1}{2}}), \tag{3.12}$$

where B_0 is a bounded constant.

4 Convergence analysis

This section is devoted to provide a convergence analysis for the numerical scheme (2.8a)-(2.8c). The goal is to show that the rate of convergence is exponential and the spectral accuracy can be obtained for the proposed approximation. Firstly, we carry out our convergence analysis in L^2 space.

Theorem 4.1. *Let y be the exact solution of the second order Volterra integro-differential equation with pantograph delay* (2.1)*, with initial condition* (2.2)*. Assume that*

$$Y^{(1)}(t) = \sum_{i=0}^{N} y_i^{(1)} F_i(t), \qquad Y(t) = \sum_{i=0}^{N} y_i F_i(t), \qquad V(t) = \sum_{i=0}^{N} v_i F_i(t),$$

where $\{y_i^{(1)}\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ and $\{v_i\}_{i=0}^N$ are given by (2.8a)-(2.8c) *and $F_i(t)$ is the i-th Lagrange basis function associated with the Legendre Gauss points $\{t_i\}_{i=0}^N$. If $y\in H^{m+1}(I)$ with $I:=(-1,1)$ for some $m\ge 1$, we have*

$$\begin{aligned} \|y^{(j)} - Y^{(j)}\|_{L^2(I)} \le & CN^{-m} \sum_{j=0}^{1} (|A_j|_{H^{m,N}(I)} + |B_j|_{H^{m,N}(I)}) \|y^{(j)}\|_{L^2(I)} \\ & + CN^{-m} \sum_{j=0}^{1} \max_{-1\le t\le 1} |K_j(t,\cdot)|_{H^{m,N}(I)} \|y^{(j)}\|_{L^2(I)} \\ & + CN^{-m} \sum_{j=0}^{1} |y^{(j)}|_{H^{m,N}(I)} + CN^{-m} |G|_{H^{m+1,N}(I)}, \end{aligned} \tag{4.1}$$

for $j=0,1$ provided that N is sufficiently large, where C is a constant independent of N.

Proof. Following the notations of (3.2b), we define

$$\big(R(s(t_i,\theta)),\phi(s(t_i,\theta))\big)_{N,t_i}=\sum_{k=0}^{N}R(s(t_i,\theta_k))\phi(s(t_i,\theta_k))w_k.$$

The numerical scheme (2.8a)-(2.8c) can be written as

$$\begin{aligned} y_i^{(1)}=&y'_{-1}+\frac{t_i+1}{2}\sum_{j=0}^{1}\Big(A_j(s(t_i,\theta)),Y^{(j)}(q_js(t_i,\theta)+q_j-1)\Big)_{N,t_i}\\ &+\frac{t_i+1}{2}\sum_{j=0}^{1}\Big(B_j(s(t_i,\theta)),Y^{(j)}(s(t_i,\theta))\Big)_{N,t_i}+\int_{-1}^{t_i}V(s)ds, \end{aligned}\tag{4.2a}$$

$$y_i=y_{-1}+\int_{-1}^{t_i}Y^{(1)}(s)ds,\tag{4.2b}$$

$$v_i=\frac{t_i+1}{2}\sum_{j=0}^{1}\Big(K_j(t_i,s(t_i,\theta)),Y^{(j)}(s(t_i,\theta))\Big)_{N,t_i}+G(t_i).\tag{4.2c}$$

In order to use Lemma 3.1, we restate Eqs. (4.2a) and (4.2c) as

$$\begin{aligned} y_i^{(1)}=&y'_{-1}+\frac{t_i+1}{2}\sum_{j=0}^{1}\int_{-1}^{1}A_j(s(t_i,\theta))Y^{(j)}(q_js(t_i,\theta)+q_j-1)d\theta\\ &+\frac{t_i+1}{2}\sum_{j=0}^{1}\int_{-1}^{1}B_j(s(t_i,\theta))Y^{(j)}(s(t_i,\theta))d\theta+\int_{-1}^{t_i}V(s)ds-\frac{t_i+1}{2}\sum_{j=0}^{1}I_j(t_i), \end{aligned}\tag{4.3a}$$

$$v_i=\frac{t_i+1}{2}\sum_{j=0}^{1}\int_{-1}^{1}K_j(t_i,s(t_i,\theta))Y^{(j)}(s(t_i,\theta))d\theta+G(t_i)-\frac{t_i+1}{2}\sum_{j=0}^{1}\widetilde{I}_j(t_i),\tag{4.3b}$$

where

$$\begin{aligned} I_j(t)=&\int_{-1}^{1}A_j(s(t,\theta))Y^{(j)}(q_js(t,\theta)+q_j-1)d\theta-\Big(A_j(s(t,\theta)),Y^{(j)}(q_js(t,\theta)+q_j-1)\Big)_{N,t}\\ &+\int_{-1}^{1}B_j(s(t,\theta))Y^{(j)}(s(t,\theta))d\theta-\Big(B_j(s(t,\theta)),Y^{(j)}(s(t,\theta))\Big)_{N,t},\\ \widetilde{I}_j(t)=&\int_{-1}^{1}K_j(t,s(t,\theta))Y^{(j)}(s(t,\theta))d\theta-\Big(K_j(t,s(t,\theta)),Y^{(j)}(s(t,\theta))\Big)_{N,t},\qquad j=0,1. \end{aligned}$$

Following from (2.6c),we have

$$\begin{aligned} y_i^{(1)}=&y'_{-1}+\sum_{j=0}^{1}\int_{-1}^{t_i}A_j(s)Y^{(j)}(q_js+q_j-1)ds+\sum_{j=0}^{1}\int_{-1}^{t_i}B_j(s)Y^{(j)}(s)ds\\ &+\int_{-1}^{t_i}V(s)ds-\frac{t_i+1}{2}\sum_{j=0}^{1}I_j(t_i), \end{aligned}\tag{4.4a}$$

$$v_i=\sum_{j=0}^{1}\int_{-1}^{t_i}K_j(t_i,s)Y^{(j)}(s)ds+G(t_i)-\frac{t_i+1}{2}\sum_{j=0}^{1}\widetilde{I}_j(t_i). \tag{4.4b}$$

Incorporating with the estimation (3.1), we have

$$\begin{aligned}&|I_j(t)|\le CN^{-m}(|A_j|_{H^{m,N}(I)}+|B_j|_{H^{m,N}(I)})\|Y^{(j)}\|_{L^2(I)},\\&|\widetilde{I}_j(t)|\le CN^{-m}|K_j(t,\cdot)|_{H^{m,N}(I)}\|Y^{(j)}\|_{L^2(I)},\qquad j=0,1.\end{aligned}$$

Multiplying $F_i(t)$ on both sides of Eqs. (4.4a), (4.2b) and (4.4b) and summing up from $i=0$ to N yield

$$\begin{aligned}Y^{(1)}(t)=&y'_{-1}+\sum_{j=0}^{1}I_N\int_{-1}^{t}A_j(s)Y^{(j)}(q_js+q_j-1)ds+\sum_{j=0}^{1}I_N\int_{-1}^{t}B_j(s)Y^{(j)}(s)ds\\&+I_N\int_{-1}^{t}V(s)ds-\sum_{j=0}^{1}J_j(t),\end{aligned} \tag{4.5a}$$

$$Y(t)=y_{-1}+I_N\int_{-1}^{t}Y^{(1)}(s)ds, \tag{4.5b}$$

$$V(t)=\sum_{j=0}^{1}I_N\int_{-1}^{t}K_j(t,s)Y^{(j)}(s)ds+I_NG(t)-\sum_{j=0}^{1}\widetilde{J}_j(t), \tag{4.5c}$$

where

$$J_j(t)=\sum_{i=0}^{N}\frac{t_i+1}{2}I_j(t_i)F_i(t),\qquad \widetilde{J}_j(t)=\sum_{i=0}^{N}\frac{t_i+1}{2}\widetilde{I}_j(t_i)F_i(t),\qquad j=0,1.$$

Similarly, multiplying $F_i(t)$ on both sides of Eqs. (2.4a)-(2.4c) and summing up from $i=0$ to N yield

$$\begin{aligned}I_Ny^{(1)}(t)=&y'_{-1}+\sum_{j=0}^{1}I_N\int_{-1}^{t}A_j(s)y^{(j)}(q_js+q_j-1)ds\\&+\sum_{j=0}^{1}I_N\int_{-1}^{t}B_j(s)y^{(j)}(s)ds+I_N\int_{-1}^{t}v(s)ds,\end{aligned} \tag{4.6a}$$

$$I_Ny(t)=y_{-1}+I_N\int_{-1}^{t}y^{(1)}(s)ds, \tag{4.6b}$$

$$I_Nv(t)=\sum_{j=0}^{1}I_N\int_{-1}^{t}K_j(t,s)y^{(j)}(s)ds+I_NG(t). \tag{4.6c}$$

It follows from (4.5a)-(4.5c) and (4.6a)-(4.6c) that

$$e_{y^{(1)}}(t)+I_N y^{(1)}(t)-y^{(1)}(t)=\sum_{j=0}^{1} I_N \int_{-1}^{t} A_j(s)e_{y^{(j)}}(q_j s+q_j-1)ds \\ +\sum_{j=0}^{1} I_N \int_{-1}^{t} B_j(s)e_{y^{(j)}}(s)ds+I_N\int_{-1}^{t} e_v(s)ds+\sum_{j=0}^{1} J_j(t), \tag{4.7a}$$

$$e_y(t)+I_N y(t)-y(t)=I_N\int_{-1}^{t} e_{y^{(1)}}(s)ds, \tag{4.7b}$$

$$e_v(t)+I_N v(t)-v(t)=\sum_{j=0}^{1} I_N \int_{-1}^{t} K_j(t,s)e_{y^{(j)}}(s)ds+\sum_{j=0}^{1}\widetilde{J}_j(t), \tag{4.7c}$$

where

$$e_{y^{(1)}}(t)=y^{(1)}(t)-Y^{(1)}(t), \qquad e_y(t)=y(t)-Y(t), \qquad e_v(t)=v(t)-V(t).$$

Consequently,

$$e_{y^{(1)}}(t)=\int_{-1}^{t} e_v(s)ds+\sum_{j=0}^{1}\int_{-1}^{t} A_j(s)e_{y^{(j)}}(q_j s+q_j-1)ds+\sum_{j=0}^{1}\int_{-1}^{t} B_j(s)e_{y^{(j)}}(s)ds \\ +\sum_{j=0}^{1} J_j(t)+f_1(t)+f_2(t)+\sum_{j=0}^{1} H_j(t), \tag{4.8a}$$

$$e_y(t)=\int_{-1}^{t} e_{y^{(1)}}(s)ds+f_3(t)+f_4(t), \tag{4.8b}$$

$$e_v(t)=\sum_{j=0}^{1}\int_{-1}^{t} K_j(t,s)e_{y^{(j)}}(s)ds+\sum_{j=0}^{1}\widetilde{J}_j(t)+f_5(t)+\sum_{j=0}^{1}\widetilde{H}_j(t), \tag{4.8c}$$

where

$$f_1(t)=y^{(1)}(t)-I_N y^{(1)}(t), \qquad f_2(t)=I_N\int_{-1}^{t} e_v(s)ds-\int_{-1}^{t} e_v(s)ds,$$

$$f_3(t)=y(t)-I_N y(t), \qquad f_4(t)=I_N\int_{-1}^{t} e_{y^{(1)}}(s)ds-\int_{-1}^{t} e_{y^{(1)}}(s)ds,$$

$$f_5(t)=v(t)-I_N v(t),$$

$$H_j(t)=I_N\int_{-1}^{t} A_j(s)e_{y^{(j)}}(q_j s+q_j-1)ds-\int_{-1}^{t} A_j(s)e_{y^{(j)}}(q_j s+q_j-1)ds \\ +I_N\int_{-1}^{t} B_j(s)e_{y^{(j)}}(s)ds-\int_{-1}^{t} B_j(s)e_{y^{(j)}}(s)ds,$$

$$\widetilde{H}_j(t)=I_N\int_{-1}^{t} K_j(t,s)e_{y^{(j)}}(s)ds-\int_{-1}^{t} K_j(t,s)e_{y^{(j)}}(s)ds, \qquad j=0,1.$$

Due to Eqs. (4.8a)-(4.8c), we use the *Dirichlet's* formula which sates

$$\int_{-1}^{t}\int_{-1}^{s}\Phi(s,\tau)d\tau ds=\int_{-1}^{t}\int_{\tau}^{t}\Phi(s,\tau)dsd\tau.$$

Provided the integral exists, we obtain

$$e_{y^{(1)}}(t)=\int_{-1}^{t}H(s,t)e_{y^{(1)}}(s)ds+\int_{-1}^{t}A_1(s)e_{y^{(1)}}(q_1s+q_1-1)ds+J(t), \tag{4.9}$$

where

$$\begin{aligned}H(s,t)=&\int_{s}^{t}\Big(\int_{s}^{\tau}K_0(\tau,z)dz\Big)d\tau+\int_{s}^{t}K_1(\tau,s)d\tau+\int_{s}^{t}B_0(\tau)d\tau+\int_{s}^{t}A_0(\tau)d\tau+B_1(s),\\ J(t)=&\int_{-1}^{t}\Big(\sum_{j=0}^{1}\widetilde{J}_j(s)+f_5(s)+\sum_{j=0}^{1}\widetilde{H}_j(s)\Big)ds+\int_{-1}^{t}\int_{-1}^{s}K_0(s,\tau)(f_3(\tau)+f_4(\tau))d\tau ds\\ &+\int_{-1}^{t}A_0(s)(f_3(q_0s+q_0-1)+f_4(q_0s+q_0-1))ds+\int_{-1}^{t}B_0(s)(f_3(s)+f_4(s))ds\\ &+\sum_{j=0}^{1}(J_j(t)+H_j(t))+f_1(t)+f_2(t).\end{aligned}$$

By (4.9), we have

$$\begin{aligned}|e_{y^{(1)}}(t)|\le&\max_{(s,t)\in\{-1\le s\le t\le 1\}}|H(s,t)|\int_{-1}^{t}|e_{y^{(1)}}(s)|ds\\ &+\max_{s\in[-1,1]}|A_1(s)|\int_{-1}^{t}|e_{y^{(1)}}(q_1s+q_1-1)|ds+|J(t)|.\end{aligned}$$

With the help of Lemma 3.4, we deduce that

$$\begin{aligned}&\|e_{y^{(1)}}(t)\|_{L^2(I)}\le C\|J(t)\|_{L^2(I)}\\ \le& C\sum_{k=1}^{5}\|f_k(t)\|_{L^2(I)}+C\sum_{j=0}^{1}\big(\|J_j(t)\|_{L^2(I)}+\|\widetilde{J}_j(t)\|_{L^2(I)}+\|H_j(t)\|_{L^2(I)}+\|\widetilde{H}_j(t)\|_{L^2(I)}\big).\end{aligned} \tag{4.10}$$

Eq. (4.8b) leads to

$$\|e_y(t)\|_{L^2(I)}\leqslant\|e_{y^{(1)}}(t)\|_{L^2(I)}+\|f_3(t)\|_{L^2(I)}+\|f_4(t)\|_{L^2(I)}. \tag{4.11}$$

We now apply Lemma 3.3 to obtain that

$$\begin{aligned}\|J_j(t)\|_{L^2(I)}\le& C\|I_j(t)\|_{L^\infty(I)}\\ \le& CN^{-m}(|A_j|_{H^{m,N}(I)}+|B_j|_{H^{m,N}(I)})(\|y^{(j)}(t)\|_{L^2(I)}+\|e_{y^{(j)}}(t)\|_{L^2(I)}),\\ \|\widetilde{J}_j(t)\|_{L^2(I)}\le& C\|\widetilde{I}_j(t)\|_{L^\infty(I)}\\ \le& CN^{-m}\max_{-1\le t\le 1}|K_j(t,\cdot)|_{H^{m,N}(I)}(\|y^{(j)}\|_{L^2(I)}+\|e_{y^{(j)}}(t)\|_{L^2(I)}),\qquad j=0,1.\end{aligned}$$

The L^2 error bounds for the interpolation polynomials (see Lemma 3.2) gives

$$
\begin{aligned}
&\|f_1(t)\|_{L^2(I)} \le CN^{-m}|y^{(1)}|_{H^{m,N}(I)},\\
&\|f_3(t)\|_{L^2(I)} \le CN^{-m}|y|_{H^{m,N}(I)},\\
&\|f_5(t)\|_{L^2(I)} \le CN^{-(m+1)}|v|_{H^{m+1,N}(I)}\\
&\qquad\qquad\quad \le CN^{-m}(|G|_{H^{m+1,N}(I)}+|y|_{H^{m,N}(I)}+|y^{(1)}|_{H^{m,N}(I)}).
\end{aligned}
$$

By virtue of (3.4a) with $m=1$,

$$
\begin{aligned}
&\|f_2(t)\|_{L^2(I)} \le CN^{-1}\|e_v(t)\|_{L^2(I)},\\
&\|f_4(t)\|_{L^2(I)} \le CN^{-1}\|e_{y^{(1)}}(t)\|_{L^2(I)},\\
&\|H_j(t)\|_{L^2(I)} \le CN^{-1}\|e_{y^{(j)}}(t)\|_{L^2(I)},\\
&\|\widetilde{H}_j(t)\|_{L^2(I)} \le CN^{-1}\Big\|K_j(t,t)e_{y^{(j)}}(t)+\int_{-1}^{t}\partial_t K_j(t,s)e_{y^{(j)}}(s)ds\Big\|_{L^2(I)}\\
&\qquad\qquad\quad \le CN^{-1}\|e_{y^{(j)}}(t)\|_{L^2(I)}, \qquad j=0,1.
\end{aligned}
$$

The above estimates, together with (4.10) and (4.11), yield

$$
\begin{aligned}
\|e_{y^{(j)}}(t)\|_{L^2(I)} \le & CN^{-m}\sum_{j=0}^{1}(|A_j|_{H^{m,N}(I)}+|B_j|_{H^{m,N}(I)})\|y^{(j)}\|_{L^2(I)}\\
&+CN^{-m}\sum_{j=0}^{1}\max_{-1\le t\le 1}|K_j(t,\cdot)|_{H^{m,N}(I)}\|y^{(j)}\|_{L^2(I)}\\
&+CN^{-m}\sum_{j=0}^{1}|y^{(j)}|_{H^{m,N}(I)}+CN^{-m}|G|_{H^{m+1,N}(I)}, \qquad (4.12)
\end{aligned}
$$

for $j=0,1$, which leads to (4.1). This completes the proof of the theorem. □

Next, we extend the L^2 error estimate to the L^∞ space. The key technique is to use an extrapolation between L^2 and H^1.

Theorem 4.2. *Let y be the exact solution of the second order Volterra integro-differential equation with pantograph delay* (2.1), *with initial condition* (2.2). *Let $Y^{(1)}(t)$, $Y(t)$ and $V(t)$ be defined in Theorem* 4.1. *If $y\in H^{m+1}(I)$ with $I:=(-1,1)$ for some $m\ge 1$, we have*

$$
\begin{aligned}
\|y^{(j)}-Y^{(j)}\|_{L^\infty(I)} \le & CN^{\frac{1}{2}-m}\sum_{j=0}^{1}(|A_j|_{H^{m,N}(I)}+|B_j|_{H^{m,N}(I)})\|y^{(j)}\|_{L^2(I)}\\
&+CN^{\frac{3}{4}-m}\sum_{j=0}^{1}|y^{(j)}|_{H^{m,N}(I)}+CN^{\frac{3}{4}-m}|G|_{H^{m+1,N}(I)}\\
&+CN^{\frac{1}{2}-m}\sum_{j=0}^{1}\max_{-1\le t\le 1}|K_j(t,\cdot)|_{H^{m,N}(I)}\|y^{(j)}\|_{L^2(I)}, \qquad (4.13)
\end{aligned}
$$

for $j=0,1$ *provided that* N *is sufficiently large, where* C *is a constant independent of* N.

Proof. Applying the inequality in the Sobolev space (see [9])

$$\|w\|_{L^\infty(a,b)} \le \Big(\frac{1}{b-a}+2\Big)^{\frac{1}{2}} \|w\|^{\frac{1}{2}}_{L^2(a,b)} \|w\|^{\frac{1}{2}}_{H^1(a,b)}, \qquad \forall w\in H^1(a,b), \tag{4.14}$$

and Lemma 3.2, we get

$$\begin{aligned}\|y-I_Ny\|_{L^\infty(I)} \le & C\|y-I_Ny\|^{\frac{1}{2}}_{L^2(I)} \|y-I_Ny\|^{\frac{1}{2}}_{H^1(I)} \\ \le & C\big(N^{-m}|y|_{H^{m,N}(I)}\big)^{\frac{1}{2}}\big(N^{\frac{3}{2}-m}|y|_{H^{m,N}(I)}\big)^{\frac{1}{2}} \\ \le & CN^{\frac{3}{4}-m}|y|_{H^{m,N}(I)}.\end{aligned} \tag{4.15}$$

Following the same procedure as in the proof of Theorem 4.1, we have the L^∞ error estimate with the help of (4.15) and Lemma 3.5. ☐

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