Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **5**, No. 2, pp. 212-221

DOI: 10.4208/aamm.12-m1257
April 2013

Some Invariant Solutions of Two-Dimensional Elastodynamics in Linear Homogeneous Isotropic Materials

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Received 20 April 2012; Accepted (in revised version) 10 October 2012

Available online 22 February 2013

Abstract. Invariant solutions of two-dimensional elastodynamics in linear homogeneous isotropic materials are considered via the group theoretical method. The second order partial differential equations of elastodynamics are reduced to ordinary differential equations under the infinitesimal operators. Three invariant solutions are constructed. Their graphical figures are presented and physical meanings are elucidated in some cases.

AMS subject classifications: 74B05, 35Q74, 22E70

Key words: Elastodynamics, group theoretical method, invariant solution.

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1 Introduction

Elastodynamics is one of the oldest topics in the theory of elasticity. It began almost 200 years ago when Navier announced the general equations of equilibrium and motion of an isotropic elastic body in 1821. However, till today, known exact solutions of elastodynamics are still very limited. In [1, 2], Kausel and Kachanov collected some exact solutions for classical and canonical problems in elastodynamics. The group theoretical method is a very powerful and versatile tool to find invariant solutions of differential equations, especially partial differential equations. It provides two basic ways: group transformation of known solutions and construction of invariant solutions. Chand [3] studied invariant solutions of one-dimensional wave propagation in dissipative materials. For a one-dimensional system of wave propagation equations in linear, viscoelastic and viscoplastic material, Ames and Suliciu constructed its invariant solutions in [4] and

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Ames obtained its group properties and conservation laws in [5]. Bokhari [6] studied the invariant solutions of a nonlinear wave equation. It is clear that the group theoretical method is also very effective in solid mechanics. The above authors studied (1+1)-dimensional problems in the space $R^{1+1}(x,t)$ by the group theoretical method. In this paper, we consider invariant solutions of (2+1)-dimensional elastodynamics in linear homogeneous isotropic materials in the space $R^{2+1}(x,y,t)$.

2 The governing equations for two-dimensional elastodynamics

Two-dimensional elastic body occupies the domain as a plane Ω. Displacement boundary is denoted by $\partial_u\Omega$. ρ is the mass density of the elastic body. λ, μ are Lame's coefficients. In the Cartesian coordinate system (x_1,x_2), displacement vector is $\mathbf{u}=(u_1,u_2)$. The theory of elastodynamics specializes in the case when all fields are time-dependent, t is time variable. In homogeneous isotropic media, the linear theory of two-dimensional elastodynamics without body force is the following equations in term of displacement $\mathbf{u}=\mathbf{u}(x_1,x_2,t)$

$$\begin{cases} (\lambda+\mu)\dfrac{\partial}{\partial x_1}\Big(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\Big)+\mu\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\Big)u_1=\rho\dfrac{\partial^2 u_1}{\partial t^2}, \\ (\lambda+\mu)\dfrac{\partial}{\partial x_2}\Big(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\Big)+\mu\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\Big)u_2=\rho\dfrac{\partial^2 u_2}{\partial t^2}. \end{cases} \tag{2.1}$$

In order to facilitate the research invariant solutions of Eq. (2.1), we now non-dimensionalize Eq. (2.1) with the characteristic length l,

$$x_i^*=\frac{x_i}{l}, \tag{2.2}$$

where i, $j=1,2$. Substituting the dimensionless quantities into Eq. (2.1) and removing the coordinate's asterisks *, Eq. (2.1) are transformed to

$$\begin{cases} \dfrac{\partial}{\partial x_1}\Big(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\Big)+\beta\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\Big)u_1=\dfrac{\partial^2 u_1}{\partial \tau^2}, \\ \dfrac{\partial}{\partial x_2}\Big(\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}\Big)+\beta\Big(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\Big)u_2=\dfrac{\partial^2 u_2}{\partial \tau^2}, \end{cases} \tag{2.3}$$

where

$$\begin{cases} \tau=t(\lambda+\mu)^{\frac{1}{2}}\rho^{-\frac{1}{2}}/l, \\ \beta=\mu/(\lambda+\mu)=1-2\nu. \end{cases} \tag{2.4}$$

In accordance with the strain energy is positive definite, we deduce the Poisson's ratio ν in the range from -1 to $1/2$. Thus the range of β is (0,3). Generally ν should be positive, because longitudinal stretching often follows transverse shrinkage. Materials with a negative Poisson's ratio are rare. Therefore, we generally take $\beta = 1/2$ in the applications.

3 Construction of invariant solutions

The group theoretical method can be used to find all symmetries of differential equations. First, it requires to solve the local Lie group of transformation admitted by the differential equations. The differential equations keep invariant under the Lie group of transformation. Specific description about the group theoretical method can be found in [7]. Infinitesimal operator admitted by Eq. (2.3) in space $R^5(x_i,u_j,\tau)$ will take the form

$$X = \xi_i \frac{\partial}{\partial x_i} + \eta_j \frac{\partial}{\partial u_j} + \omega \frac{\partial}{\partial \tau}, \tag{3.1}$$

where ξ_i, η_j, ω are unknown functions of x_i, u_j, τ. Repeated indices mean summation over the indices.

Partial derivatives of function displacements u_j with respect to x_i, τ in Eq. (2.3) have only second order

$$\frac{\partial^2 u_1}{\partial x_1^2},\ \frac{\partial^2 u_1}{\partial x_2^2},\ \frac{\partial^2 u_2}{\partial x_1^2},\ \frac{\partial^2 u_2}{\partial x_2^2},\ \frac{\partial^2 u_1}{\partial x_2 \partial x_1},\ \frac{\partial^2 u_2}{\partial x_1 \partial x_2},\ \frac{\partial^2 u_1}{\partial \tau^2},\ \frac{\partial^2 u_2}{\partial \tau^2}. \tag{3.2}$$

We need to extend the space $R^5(x_i,u_j,\tau)$ twice. The first extended space will be $R^{11}(x_i,u_j,\tau,u_{j,i},u_{j,\tau})$. The second extended space will be $R^{21}(x_i,u_j,\tau,u_{j,i},u_{j,ki},u_{j,\tau},u_{j,\tau\tau})$, i, j, $k=1,2$. Let us denote

$$\begin{cases} x_3 = \tau, \\ p_i^j = u_{j,i} = \dfrac{\partial u_j}{\partial x_i}, \\ r_{ki}^j = u_{j,ki} = \dfrac{\partial^2 u_j}{\partial x_k \partial x_i}. \end{cases} \tag{3.3}$$

The first extended infinitesimal operator of the operator (3.1) is

$$\begin{cases} pr^{(1)} X = X + \zeta_n^j \dfrac{\partial}{\partial p_n^j}, \\ \zeta_n^j = D_n(\eta_j) - p_m^j D_n(\xi_m), \\ D_n = \dfrac{\partial}{\partial x_n} + p_n^i \dfrac{\partial}{\partial u_i}. \end{cases} \tag{3.4}$$

The second extended infinitesimal operator of the operator (3.1) is

$$\begin{cases} pr^{(2)}X = pr^{(1)}X + \sigma^j_{mn}\dfrac{\partial}{\partial r^j_{mn}}, \\ \sigma^j_{mn} = \tilde{D}_n(\zeta^j_m) - r^j_{m\phi}\tilde{D}_n(\xi_\phi), \\ \tilde{D}_n = \dfrac{\partial}{\partial x_n} + p^i_n\dfrac{\partial}{\partial u_i} + r^i_{mn}\dfrac{\partial}{\partial p^i_m}, \end{cases} \tag{3.5}$$

where i, j, $k=1,2$; m, n, $\phi=1,2,3$. Eqs. (3.4) and (3.5) exclude mixed partial derivatives of displacements u_j with respect to coordinates x_i and time τ.

According to the invariability condition of the group theoretical method, we obtain

$$\begin{cases} pr^{(2)}\Big[\dfrac{\partial}{\partial x_1}\Big(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2}\Big) + \beta\Big(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\Big)u_1 - \dfrac{\partial^2 u_1}{\partial \tau^2}\Big] = 0, \\ pr^{(2)}\Big[\dfrac{\partial}{\partial x_2}\Big(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2}\Big) + \beta\Big(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\Big)u_2 - \dfrac{\partial^2 u_2}{\partial \tau^2}\Big] = 0, \end{cases} \tag{3.6}$$

viz.,

$$\begin{cases} (1+\beta)\sigma^1_{11} + \sigma^2_{12} + \beta\sigma^1_{22} - \sigma^1_{33} = 0, \\ (1+\beta)\sigma^2_{22} + \sigma^1_{21} + \beta\sigma^2_{11} - \sigma^2_{33} = 0, \end{cases} \tag{3.7}$$

where

$$\begin{aligned} \sigma^\alpha_{mn} = &\Big(\frac{\partial}{\partial x_n} + p^k_n\frac{\partial}{\partial u_k} + r^k_{qn}\frac{\partial}{\partial p^k_q}\Big)\Big[\Big(\frac{\partial}{\partial x_m} + p^\varphi_m\frac{\partial}{\partial u_\varphi}\Big)\eta_\alpha - p^\alpha_d\Big(\frac{\partial}{\partial x_m} + p^\varphi_m\frac{\partial}{\partial u_\varphi}\Big)\xi_d\Big] \\ &- r^\alpha_{m\beta}\Big(\frac{\partial}{\partial x_n} + p^k_n\frac{\partial}{\partial u_k} + r^k_{qn}\frac{\partial}{\partial p^k_q}\Big)\xi_\beta. \end{aligned} \tag{3.8}$$

Solving Eq. (3.9), we can get the following infinitesimal operator admitted by Eq. (2.3)

$$\begin{aligned} X =& \xi_i\frac{\partial}{\partial x_i} + \eta_j\frac{\partial}{\partial u_j} + \omega\frac{\partial}{\partial \tau} \\ =& (c_1 + c_4x_1 - c_5x_2)\frac{\partial}{\partial x_1} + (c_2 + c_4x_2 + c_5x_1)\frac{\partial}{\partial x_2} \\ &+ (-c_5u_2 + c_6u_1 + c_7s_1)\frac{\partial}{\partial u_1} + (c_5u_1 + c_6u_2 + c_7s_2)\frac{\partial}{\partial u_2} + (c_3 + c_4t)\frac{\partial}{\partial \tau}. \end{aligned} \tag{3.9}$$

Where c_i $(i=1,\cdots,7)$ are some integral constants.

The base of the Lie algebra L corresponding to the Lie group G admitted by Eq. (2.3) is

$$\begin{cases} X_1 = \dfrac{\partial}{\partial x_1}, \quad X_2 = \dfrac{\partial}{\partial x_2}, \quad X_3 = \dfrac{\partial}{\partial \tau}, \\ X_4 = x_1 \dfrac{\partial}{\partial x_1} + x_2 \dfrac{\partial}{\partial x_2} + \tau \dfrac{\partial}{\partial \tau}, \\ X_5 = x_1 \dfrac{\partial}{\partial x_2} - x_2 \dfrac{\partial}{\partial x_1} + u_1 \dfrac{\partial}{\partial u_2} - u_2 \dfrac{\partial}{\partial u_1}, \end{cases} \tag{3.10}$$

and

$$\begin{cases} Y_1 = u_1 \dfrac{\partial}{\partial u_1} + u_2 \dfrac{\partial}{\partial u_2}, \\ Y_\infty = s_1 \dfrac{\partial}{\partial u_1} + s_2 \dfrac{\partial}{\partial u_2}, \end{cases} \tag{3.11}$$

where $\mathbf{s} = (s_1, s_2)$ is an arbitrary solution of Eq. (2.3).

The infinitesimal operators in (3.10) determine a five dimensional Lie algebra L_5 corresponding to a Lie group G_5. The infinitesimal operators in (3.11) determine an infinite Lie algebra L_∞ corresponding to a normal subgroup N of the Lie group G. Here, we provide types of transformation corresponding base of the Lie algebra L_5. The results are shown in Table 1.

Table 1: Types of transformation corresponding base of the Lie algebra L_5.

Operators	Types of transformation	Physical explanation
X_1, X_2, X_3	$x_1^* = x_1 + a_1$, $x_2^* = x_2 + a_2$, $\tau^* = \tau + a_3$	Translation transformations
X_4	$x_1^* = x_1 \exp a_4$, $x_2^* = x_2 \exp a_4$, $\tau^* = \tau \exp a_4$	Stretching transformation
X_5	$x_1^* = x_1 \cos a_5 + x_2 \sin a_5$, $x_2^* = -x_1 \sin a_5 + x_2 \cos a_5$ $u_1^* = u_1 \cos a_5 + u_2 \sin a_5$, $u_2^* = -u_1 \sin a_5 + u_2 \cos a_5$	Rotation transformation

We can be taken **s** as a general solution in the theory of elastodynamics, e.g., Green-Lame solution, Cauchy-Kovalevski-Somigliana solution, Naghdi-Hsu type solution and Boussinesq-Papkovitch-Neuber solution in [8]. Thus, according to the properties of the group theoretical method, general solution also is a kind of invariant solutions under some infinitesimal operators or combined infinitesimal operators.

We shall solve several invariant solutions below with the help of the Lie algebra L_5 admitted by the differential equations (2.3).

a) The infinitesimal operator

$$X_4 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \tau \frac{\partial}{\partial \tau}$$

gives invariant

$$J = \frac{x_1}{\tau}.$$

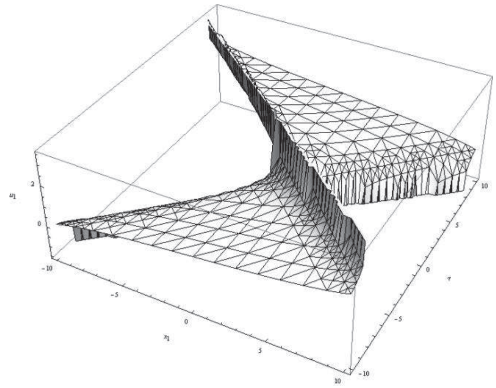

Figure 1: The solution surface of u_1 for $\beta=1/2$.

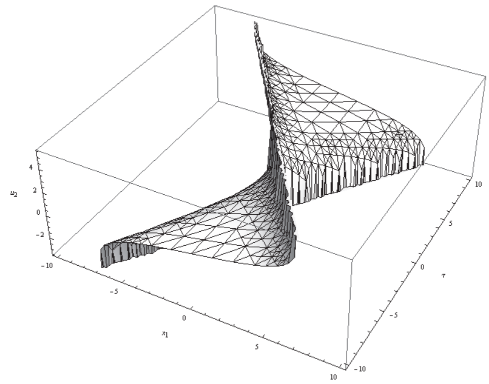

Figure 2: The solution surface of u_2 for $\beta=1/2$.

Under this infinitesimal operator, invariant solution will take the form

$$\begin{cases} u_1=f(J), \\ u_2=g(J). \end{cases} \tag{3.12}$$

Substituting the invariant solution into Eqs. (2.3), we arrive at the following equations

$$\begin{cases} (J^2-\beta-1)\dfrac{d^2f}{dJ^2}+2J\dfrac{df}{dJ}=0, \\ (J^2-\beta)\dfrac{d^2g}{dJ^2}+2J\dfrac{dg}{dJ}=0. \end{cases} \tag{3.13}$$

These (3.13) are second order ordinary differential equations. Let us integrate the above equation and obtain

$$\begin{cases} u_1=f(x_1,\tau)=\dfrac{c_1}{2\sqrt{\beta+1}}\ln\dfrac{(\sqrt{\beta+1}-x_1/\tau)}{(\sqrt{\beta+1}+x_1/\tau)}+c_2, \\ u_2=g(x_1,\tau)=\dfrac{c_3}{2\sqrt{\beta}}\ln\dfrac{(\sqrt{\beta}-x_1/\tau)}{(\sqrt{\beta}+x_1/\tau)}+c_4, \end{cases} \tag{3.14}$$

where c_1, c_2, c_3, c_4 are integral constants.

The graphical figures of displacements f, g corresponding to $c_i=1$; $\beta=1/2$ are shown in Figs. 1 and 2.

b) Let us denote κ is an arbitrary constant. The combined infinitesimal operator κX_3+X_1 and infinitesimal operator X_2 give invariant $J=\tau-\kappa x_1$. Under these infinitesimal operators, invariant solution will take the form

$$\begin{cases} u_1=f(J), \\ u_2=g(J). \end{cases} \tag{3.15}$$

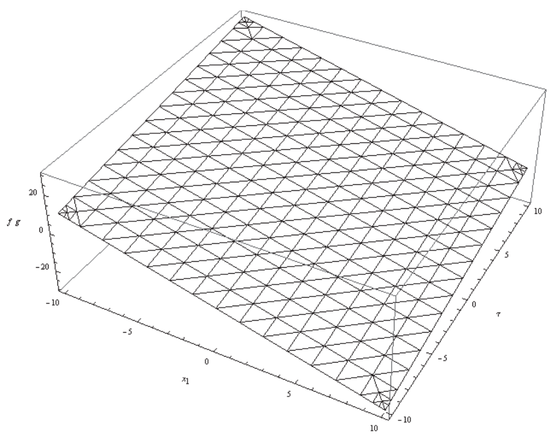

Figure 3: The solution surface of u_1 for $\kappa=2$.

Substituting the invariant solution into Eq. (2.3), we arrive at the following equations

$$\begin{cases} [(1+\beta)\kappa^2-1]\dfrac{d^2f}{dJ^2}=0, \\ (\beta\kappa^2-1)\dfrac{d^2g}{dJ^2}=0. \end{cases} \tag{3.16}$$

When $\kappa^2\neq 1/(1+\beta)$ and $\kappa^2\neq 1/\beta$, we integrate Eq. (3.16) and obtain

$$\begin{cases} u_1=f=c_1(\tau-\kappa x_1)+c_2, \\ u_2=g=c_1(\tau-\kappa x_1)+c_2, \end{cases} \tag{3.17}$$

where c_1, c_2, c_3, c_4 are integral constants. This invariant solution is the solution of one-dimensional wave propagation equations, namely traveling wave solution. According to the solution (3.17), wave displacement field has nothing to do with the media. The graphical figure of displacement u_1 correspond to $c_i=1$; $\kappa=2$ is shown in Fig. 3. The graphical figure of displacement u_2 is similar in form to displacement u_1.

c) The combined infinitesimal operator κX_3+X_1 and infinitesimal operator X_4 give invariant $J=(\tau-\kappa x_1)/x_2$. Under these infinitesimal operators, invariant solution will take the form

$$\begin{cases} u_1=a_1F(J)+a_2G(J), \\ u_2=a_2F(J)+a_1G(J), \end{cases} \tag{3.18}$$

where a_1, a_2 are arbitrary constants.

The invariant solution are substituted into Eq. (2.3), thus

$$G(\chi)=-\frac{\kappa}{2}\ln\frac{\beta[(\tau-\kappa x_1)^2/x_2^2+\kappa^2]-1}{(1+\beta)[(\tau-\kappa x_1)^2/x_2^2+\kappa^2]-1}, \tag{3.19a}$$

$$F(\chi)=\begin{cases} x_2/(\tau-\kappa x_1), & \beta\kappa^2=1,\\ \arctan\sqrt{\beta}(\tau-\kappa x_1)/(x_2\sqrt{\beta\kappa^2-1}), & \beta\kappa^2>1,\\ \dfrac{1}{2}\ln\dfrac{(\tau-\kappa x_1)/x_2-\sqrt{\beta}/\sqrt{1-\beta k^2}}{(\tau-\kappa x_1)/x_2+\sqrt{\beta}/\sqrt{1-\beta k^2}}, & \beta\kappa^2<1. \end{cases} \tag{3.19b}$$

Here, we show that invariant solution $\mathbf{u}(x_1,x_2,\tau)$ is a stationary self-similar solution when $\tau=0$. The graphical figures of displacements u_1 correspond to $a_1=a_2=1$; $\tau=1$; $\beta=1/2$; $\kappa=\sqrt{2}$,2,0.5 are shown in Figs. 4, 5 and 6.

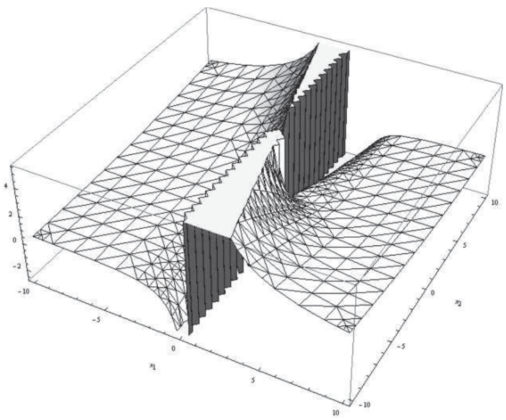

Figure 4: The solution surface of u_1 for $\beta\kappa^2=1$.

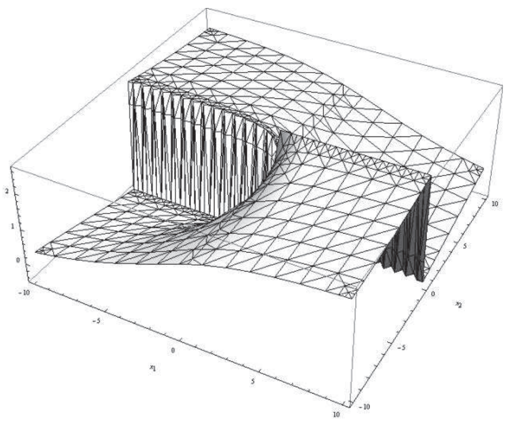

Figure 5: The solution surface of u_1 for $\beta\kappa^2>1$.

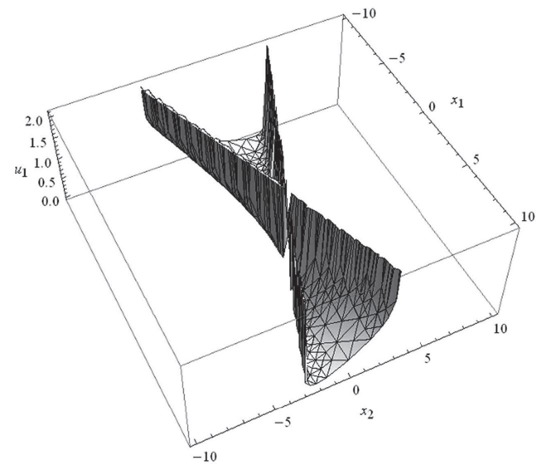

Figure 6: The solution surface of u_1 for $\beta\kappa^2 < 1$.

Figs. 4, 5 and 6 showed discontinuities of the invariant solutions. Regarding this, we hereby make the following explanations. In continuum mechanics, the displacement field $\mathbf{u}$ will be a continuous function of the coordinate vector $\mathbf{x}$ at any time and no fragmentation and overlap of media micelle. As we known, sudden jump or discontinuity of some physical variables on both sides of the shock surface in fluid mechanics, there may be one or more discontinuity surfaces in solid mechanics, their physical variables also have some discontinuities on both sides of the discontinuity surface. For example, an impulse load acts on the plate side and it is parallel to middle plane of the plate. In the plate, stress, strain and displacement fields show some discontinuities in wave fronts. The plate is divided into two regions by the shock wave. But stress, strain and displacement fields are continuous in each region. It should be noted, this analogy also has its limits, because a quasilinear PDE governs the system in fluid mechanics, but in the current setting, we considered a fully linear PDE.

4 Concluding remarks

Construction of solutions is effective one of the ways to find invariant solutions of differential equations. In this work, we have shown how to construct invariant solutions of two-dimensional elastodynamics using the group theoretical method. We know that the governing equations for two-dimensional elastodynamics are second order partial differential equations. The symmetries of elastodynamics, via their infinitesimal operators, reduce the governing equations to ordinary differential equations. Difficulty in solving the governing equations is lowered. We construct some invariant solutions of two-dimensional elastodynamics in linear homogeneous isotropic materials, present their graphical figures, and elucidate physical meanings of invariant solutions in some cases. For example, the invariant solution under operators $\kappa X_3 + X_1$ and X_2 is a traveling wave solution.

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