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Galerkin Formulations of the Method of Fundamental Solutions

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Dedicated to Graeme Fairweather on the occasion of his 70^{th} *birthday.*

Abstract. In this paper, we introduce two Galerkin formulations of the Method of Fundamental Solutions (MFS). In contrast to the collocation formulation of the MFS, the proposed Galerkin formulations involve the evaluation of integrals over the boundary of the domain under consideration. On the other hand, these formulations lead to some desirable properties of the stiffness matrix such as symmetry in certain cases. Several numerical examples are considered by these methods and their various features compared.

AMS subject classifications: 65N38, 65N30, 65N35, 65N80

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1 Introduction

The method of fundamental solutions (MFS) was introduced as a numerical method in the late seventies in a paper by Mathon and Johnston [18], followed shortly afterwards by applications to potential problems in papers by Fairweather and Johnston [6, 14]. Since then it has been applied to a wide range of problems in engineering science [2, 5, 7, 8, 21]. In the MFS, the approximate solution is taken as a linear superposition of singular solutions (fundamental solutions or Green's functions) of the differential operator for the problem of interest. As such, the approximate solution will satisfy the governing

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differential equation for the problem provided all singular points are located outside the domain of interest. In the traditional MFS approach, the coefficients in the MFS expansion are determined through either a linear or non-linear least-squares formulation using collocation. The linear approach utilizes source locations exterior to the domain of interest which are fixed *a priori*. In contrast, the nonlinear approach determines the source locations (also required to be exterior to the domain of interest) as part of the solution. Collocation points are selected on the physical boundary where the boundary conditions for the MFS approximations are minimized. It should be mentioned that the theoretical aspects of a Galerkin MFS for the solution of exterior Helmholtz problems were studied in [16] and a Galerkin-type MFS for harmonic problems was proposed in [9] but, to the best of our knowledge, there has been no follow-up since.

One problem with the strong approximate solution obtained by collocation methods is that one usually expects significant error to occur between the collocation points. One way to circumvent this issue is by employing a Galerkin approach where the boundary residuals are minimized in an average sense over the entire boundary instead of just at the collocation points. Galerkin approaches may also be employed in boundary element formulations (and other formulations) to also obtain symmetric coefficient matrices [17, 22].

The MFS has its origins in Trefftz's methods [15,23] which were originally developed as an alternative approach to Ritz's method for approximating the solution of partial differential equations. The primary difference between the two methods is that Trefftz's methods rely on the use of nonsingular basis functions which form a complete set of solutions to the differential equation, while the MFS utilizes singular fundamental solutions. It is worth noting that the approach of superposing singular solutions has been used for some time, see, for example, the application of singular source superposition described in [10] for problems in elastostatics. Trefftz methods can be developed from an indirect boundary integral equation [11] and the resulting minimization problem for the source strengths can be formulated for numerical computation either via a collocation approach or a Galerkin method [12]. A weighted residual approach leading to a Trefftz boundary element approach has also been discussed [19,20] where both collocation and Galerkin methods were used for the numerical solution of the problems considered. Interestingly, in [19] the authors found that both the collocation and Galerkin approaches yielded about the same degree of accuracy in the computations. It is also argued that the Galerkin approach is more economical since the resulting linear system is smaller than the one obtained with collocation. On the other hand, the Galerkin method requires numerical integration where collocation does not.

Our paper is organized as follows. After presenting the general formulation for the MFS in Section 2, we next detail the usual collocation formulation in Section 3 where implementation is also discussed. In sections 4 and 5 we present two alternative formulations for the Galerkin MFS and discuss implementation details. Symmetric coefficient matrices can be obtained from the Galerkin formulations presented and details concerning this symmetry are addressed in the Appendix. We finally discuss the Dirichlet prob-