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The Crank-Nicolson Hermite Cubic Orthogonal Spline Collocation Method for the Heat Equation with Nonlocal Boundary Conditions

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Dedicated to Professor Graeme Fairweather in honor of his 70^{th} Birthday.

Abstract. We formulate and analyze the Crank-Nicolson Hermite cubic orthogonal spline collocation method for the solution of the heat equation in one space variable with nonlocal boundary conditions involving integrals of the unknown solution over the spatial interval. Using an extension of the analysis of Douglas and Dupont [23] for Dirichlet boundary conditions, we derive optimal order error estimates in the discrete maximum norm in time and the continuous maximum norm in space. We discuss the solution of the linear system arising at each time level via the capacitance matrix technique and the package COLROW for solving almost block diagonal linear systems. We present numerical examples that confirm the theoretical global error estimates and exhibit superconvergence phenomena.

AMS subject classifications: 65N35, 65N12, 65N15

Key words: Heat equation, nonlocal boundary conditions, orthogonal spline collocation, Hermite cubic splines, convergence analysis, superconvergence.

1 Introduction

Consider the heat equation

$$u_t - u_{xx} = f(x,t), \quad x \in [0,1], \quad t \in [0,T], \tag{1.1}$$

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subject to the initial condition

$$u(x,0)=g(x),\quad x\in[0,1], \tag{1.2}$$

and the nonlocal boundary conditions

$$u(0,t)=\int_0^1\alpha(x)u(x,t)dx+g_0(t),\quad u(1,t)=\int_0^1\beta(x)u(x,t)dx+g_1(t),\quad t\in[0,T], \tag{1.3}$$

where $\alpha,\beta\in C[0,1]$ and

$$\|\alpha\|_{L^1(0,1)}<1,\quad \|\beta\|_{L^1(0,1)}<1. \tag{1.4}$$

It is shown in [14] that such problems arise in thermoelasticity. The existence, uniqueness and properties of solutions, even in several space variables, have been studied in [14, 15, 28, 31].

Finite difference methods have been used frequently for the numerical solution of (1.1)-(1.3). One of the first was that of Wang and Lin [48] who formulated a method based on the Crank-Nicolson (CN) method with Simpson's rule to approximate the integrals in (1.3). No analysis was provided. Ekolin [24] considered the forward and backward Euler methods and the CN method, each with the trapezoidal rule for the approximation of the integrals, and derived error estimates for all three methods. Ekolin's analysis of the CN method requires the condition

$$\|\alpha\|_{L^2(0,1)}+\|\beta\|_{L^2(0,1)}<\frac{\sqrt{3}}{2}, \tag{1.5}$$

in addition to (1.4). Liu [36] considered θ-methods with $\theta\ge 1/2$ and derived error estimates with (1.5) replaced by the weaker condition

$$\|\alpha\|^2_{L^2(0,1)}+\|\beta\|^2_{L^2(0,1)}<2. \tag{1.6}$$

In [43], Pan provided analyses of the forward and backward Euler methods without constraints on the functions α and β. Sun [47] derived a method which is fourth-order accurate in space and second-order in time under the condition

$$\|\alpha\|_{L^2(0,1)}+\|\beta\|_{L^2(0,1)}<\sqrt{0.432}. \tag{1.7}$$

This method is based on the high-order method (HOM) of Douglas [21] (which is not referenced in [47]) together with Simpson's rule for the approximation of the integrals. In [40], the method claimed by Dehghan [17] to be fourth-order in space is shown to be only second-order, and the correct fourth-order method is derived, a method which is similar to that of Sun [47]. In [17, 40], this implicit method is called Crandall's method when in fact it is also based on the HOM of Douglas of which the method of Crandall [13] is the special case in which a specific value of the mesh ratio yields an explicit method. A