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Alternating Direction Implicit Orthogonal Spline Collocation on Non-Rectangular Regions

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Dedicated to Professor Graeme Fairweather in honor of his 70th birthday.

Abstract. The alternating direction implicit (ADI) method is a highly efficient technique for solving multi-dimensional time dependent initial-boundary value problems on rectangles. When the ADI technique is coupled with orthogonal spline collocation (OSC) for discretization in space we not only obtain the global solution efficiently but the discretization error with respect to space variables can be of an arbitrarily high order. In [2], we used a Crank Nicolson ADI OSC method for solving general nonlinear parabolic problems with Robin's boundary conditions on rectangular polygons and demonstrated numerically the accuracy in various norms. A natural question that arises is: Does this method have an extension to non-rectangular regions? In this paper, we present a simple idea of how the ADI OSC technique can be extended to some such regions. Our approach depends on the transfer of Dirichlet boundary conditions in the solution of a two-point boundary value problem (TPBVP). We illustrate our idea for the solution of the heat equation on the unit disc using piecewise Hermite cubics.

AMS subject classifications: 65M70

Key words: Alternating direction implicit method, orthogonal spline collocation, two point boundary value problem, Crank Nicolson, parabolic equation, non-rectangular region.

1 Introduction

ADI methods were first introduced in the context of finite differences (FD) by Peaceman and Rachford [9] for solving elliptic and parabolic differential equations. Over the past

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70 years the ADI technique has been coupled with finite element Galerkin (FEG) and OSC discretizations in space for solving efficiently a variety of multi-dimensional time dependent initial-boundary value problems on rectangles, see [7] and references therein for a brief overview of ADI FEG and ADI OSC methods.

For over 20 years, we have been developing and analysing new ADI OSC methods for solving linear and nonlinear time dependent problems on rectangles and rectangular polygons. In [1], we formulated and analyzed a Crank Nicolson ADI OSC method for the solution of general linear parabolic problems with Dirichlet boundary conditions on rectangles. In [2], we formulated a Crank Nicolson ADI OSC method to solve general nonlinear parabolic problems with Robin's boundary conditions on rectangular polygons. The merits of these schemes and comparisons with ADI FD and ADI FEG methods have been discussed in [7]. A natural question that arises is: Does the ADI OSC technique have an extension to non-rectangular regions such as a triangle, a disc, a quadrilateral, etc? The purpose of this paper is to present a simple idea of how the ADI OSC technique can be extended to some such regions. Our approach depends on the transfer of Dirichlet boundary conditions in the solution of TPBVP on the original interval without the end subintervals of a non-uniform partition. We present our idea for the solution of the heat equation on the unit disc in space using piecewise Hermite cubics in the space coordinate directions.

A brief outline of the paper is as follows. In Section 2 we consider solution of TPBVP on the original interval without the end subintervals. In Section 3 the ADI OSC method for a unit disc is explained and numerical results, demonstrating the optimal rate of convergence in the maximum norm, are presented. Concluding remarks are given in Section 4.

2 OSC for TPBVP without end subintervals

Consider the TPBVP on $[a,b]$ with Dirichlet boundary conditions

$$Lu=f(x), \quad x\in(a,b), \quad u(a)=u_a, \quad u(b)=u_b, \tag{2.1}$$

where a, b, u_a, u_b are given numbers, $a<b$, f is a given function on (a,b), and, with r a given nonnegative function on (a,b),

$$Lu=-u''+r(x)u. \tag{2.2}$$

Assume that N is a natural number, $\{x_i\}_{i=0}^N$ is, in general, a nonuniform partition of $[a,b]$, that is,

$$a=x_0<x_1<\cdots<x_{N-1}<x_N=b.$$

Observe that there are N subintervals corresponding to the partition $\{x_i\}_{i=0}^N$. We want to approximate u of (2.1)-(2.2) on $[x_1,x_{N-1}]$ rather than $[x_0,x_N]$; see Fig. 1.