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# *B*-Spline Gaussian Collocation Software for Two-Dimensional Parabolic PDEs

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**Abstract.** In this paper we describe new *B*-spline Gaussian collocation software for solving two-dimensional parabolic partial differential equations (PDEs) defined over a rectangular region. The numerical solution is represented as a bi-variate piecewise polynomial (using a tensor product *B*-spline basis) with time-dependent unknown coefficients. These coefficients are determined by imposing collocation conditions: the numerical solution is required to satisfy the PDE and boundary conditions at images of the Gauss points mapped onto certain subregions of the spatial domain. This leads to a large system of time-dependent differential algebraic equations (DAEs) which is solved using the DAE solver, DASPK. We provide numerical results in which we use the new software, called BACOL2D, to solve three test problems.

**AMS subject classifications**: 65M20, 65M70

**Key words**: Collocation, *B*-splines, two-dimensional partial differential equations, differential-algebraic equations.

## 1 Introduction

In this paper, we discuss a numerical algorithm that uses high order methods in time and space to solve a system of $n$ parabolic partial differential equations (PDEs) in two space dimensions. We assume a problem class having the form

$$u_t(x,y,t)=f\big(x,y,t,u(x,y,t),u_x(x,y,t),u_y(x,y,t),u_{xx}(x,y,t),u_{xy}(x,y,t),u_{yy}(x,y,t)\big), \quad (1.1)$$

for $(x,y,t)\in\Omega\times(t_0,t_{out}]$, where $\Omega=\{(x,y)|a<x<b,\ c<y<d\}$ and $u(x,y,t)$ is a vector function with $n$ components. The boundary conditions at $x=a$ and $x=b$ are

$$g_a(y,t,u(a,y,t),u_x(a,y,t),u_y(a,y,t))=0, \quad g_b(y,t,u(b,y,t),u_x(b,y,t),u_y(b,y,t))=0,$$

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for $t\in(t_0,t_{out}]$, while the boundary conditions at $y=c$ and $y=d$ are

$$g_c(x,t,u(x,c,t),u_x(x,c,t),u_y(x,c,t))=0, \quad g_d(x,t,u(x,d,t),u_x(x,d,t),u_y(x,d,t))=0,$$

for $t\in(t_0,t_{out}]$. The initial conditions at $t=t_0$ are given by

$$u(x,y,t_0)=u_0(x,y), \quad (x,y)\in\Omega\cup\partial\Omega.$$

In the above, $f$, $g_a$, $g_b$, $g_c$, $g_d$ and $u_0$ are given vector function with $n$ components.

The numerical approach we employ uses two-dimensional (2D) $B$-spline Gaussian collocation to simultaneously discretize the $x$ and $y$ directions. The approximate solution is represented as a bi-variate piecewise polynomial (of degree $p$ in $x$ and degree $q$ in $y$ where $3\le p,q\le 11$) implemented in terms of a tensor product $B$-spline basis [3], with time-dependent unknown coefficients. We require the approximate solution to satisfy the PDE and boundary conditions at images of the Gauss points mapped onto certain subregions of the spatial domain, and this leads to a system of differential algebraic equations (DAEs). Since this DAE system is usually somewhat large, we use the DAE solver, DASPK [5], which is designed to efficiently solve large scale DAEs. DASPK uses a family of Backward Differentiation Formulas (BDFs) of orders 1 to 5 for the time integration. Our implementation of this approach is called BACOL2D. The BACOL2D software is a generalization of the software package, BACOL [30–32], designed for the numerical solution of systems of one-dimensional (1D) parabolic PDEs.

In Section 2, we provide a brief review of the related literature, focusing on work that features the use of collocation methods for 2D PDEs. We also review, in detail, the BACOL package since the 1D $B$-spline Gaussian collocation algorithm it employs is the basis for the 2D $B$-spline Gaussian collocation algorithm we consider in this paper. Section 3 discusses the BACOL2D implementation; this involves a description of the 2D $B$-spline Gaussian collocation algorithm, a discussion of the use of the DASPK package to solve the large DAE system arising from the collocation spatial discretization, and a description of a fast block LU algorithm for the treatment of certain structured matrices that arise during the computation. In Section 4 we present numerical results obtained by using BACOL2D to solve several test problems; these results allow us to experimentally demonstrate the order of convergence of the 2D collocation solutions. Section 5 provides our summary and identifies some areas for future work.

## 2 Background

### 2.1 Collocation methods for 2D PDEs

There is of course a very large body of literature on the numerical solution of PDEs–see, e.g., the recently published research texts [11, 18–20, 22] and references within. Here we focus on literature that considers collocation methods for 2D PDEs.