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A New Composite Quadrature Rule

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Dedicated to Professor Graeme Fairweather on the occasion of his 70^{th} birthday.

Abstract. We present a new composite quadrature rule which is exact for polynomials of degree $2N+K-1$ with N abscissas at each subinterval and K boundary conditions. The corresponding orthogonal polynomials are introduced and the analytic formulae for abscissas and weight functions are presented. Numerical results show that the new quadrature rule is more efficient, compared with classical ones.

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1 Introduction

The Gaussian quadrature rule is an interpolatory quadrature rule on zeros of certain orthogonal polynomials. A general form can be given by

$$\int_{-1}^{1} f(x)dx=\sum_{j=1}^{N}\omega_j f(x_j)+R_f$$

and its composite rule is

$$\int_a^b f(t)dt=\sum_{i=0}^{M-1} h\Big(\sum_{j=1}^{N}\omega_j f\Big(t_i+\frac{h}{2}+x_j\frac{h}{2}\Big)\Big)+\mathcal{O}(h^{2N}) \tag{1.1}$$

for a uniform partition $(h=t_{i+1}-t_i)$. The Gaussian quadrature rule is exact for polynomials of degree no larger than $2N-1$. The composite Gaussian quadrature rule is one of

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most efficient and commonly used methods for numerical integration [2,4,7,8,13], particularly for problems in which the integrand is defined piecewisely. The computational complexity of Gauss-type rules depends upon the number of points where the integrand is evaluated. The composite Gaussian quadrature rule needs to evaluate the integrand $f(x)$ at NM points.

Here we propose a new quadrature rule of the form

$$\int_{-1}^{1} f(x)dx \approx \sum_{j=1}^{N}\omega_j f(x_j)+\sum_{i=1}^{K}\beta_i\big(f^{(i-1)}(1)-f^{(i-1)}(-1)\big), \tag{1.2}$$

where ω_j, x_j and β_i are to be determined so that the formula is exact for any polynomials of degree no larger than $2N+K-1$. The corresponding composite rule is given by

$$\int_a^b f(t)dt=\sum_{i=0}^{M-1}\Big(\sum_{j=1}^{N}\frac{h}{2}\omega_j f\Big(t_i+\frac{h}{2}+x_j\frac{h}{2}\Big)\Big)+\sum_{i=1}^{K}\Big(\frac{h}{2}\Big)^i\beta_i\big(f^{(i-1)}(b)-f^{(i-1)}(a)\big)+R_f \tag{1.3}$$

with $R_f=\mathcal{O}(h^{2N+K})$ in general.

There are several interesting applications. For $K=1$, (1.3) becomes

$$\int_a^b f(x)dx=\sum_{i=0}^{M-1}\Big(\sum_{j=1}^{N}\frac{h}{2}\omega_j f\Big(t_i+\frac{h}{2}+x_j\frac{h}{2}\Big)\Big)+\frac{h}{2}\beta_1\big(f(b)-f(a)\big)+\mathcal{O}(h^{2N+1}). \tag{1.4}$$

This composite quadrature rule only needs two extra evaluations of the integrand at endpoints. The error of this rule is one order higher than the classical one. The new rule is more significant for those small N, which are often used in practical computations. We shall show that the weight $\omega_j>0$ and the new rule is stable although it is not a positive Gaussian quadrature.

When the integrand $f(t)$ satisfies some periodic conditions

$$f^{(k-1)}(b)=f^{(k-1)}(a),\quad k=1,2,\cdots,K, \tag{1.5}$$

we have

$$\int_a^b f(t)dt\approx\sum_{i=0}^{M-1}\Big(\sum_{j=1}^{N}\alpha_j f\Big(t_i+\frac{h}{2}+x_j\frac{h}{2}\Big)\Big)+\mathcal{O}(h^{2N+K}). \tag{1.6}$$

For more general case, one can use the classical sigmoidal transformation or IMI transformation [2,3], which changes the integrand $f(x)$ into one satisfying some periodic conditions in (1.5).

A similar quadrature formula is the Gaussian-Lobatto rule, see [3] given in the form of

$$\int_{-1}^{1} f(x)dx=\sum_{j=1}^{N}\omega_j f(x_j)+\sum_{i=1}^{K}\beta_i\big(f^{(i-1)}(1)+(-1)^i f^{(i-1)}(-1)\big)+R_f.$$