

Galerkin Formulations of the Method of Fundamental Solutions

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Dedicated to Graeme Fairweather on the occasion of his 70th birthday.

Abstract. In this paper, we introduce two Galerkin formulations of the Method of Fundamental Solutions (MFS). In contrast to the collocation formulation of the MFS, the proposed Galerkin formulations involve the evaluation of integrals over the boundary of the domain under consideration. On the other hand, these formulations lead to some desirable properties of the stiffness matrix such as symmetry in certain cases. Several numerical examples are considered by these methods and their various features compared.

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1 Introduction

The method of fundamental solutions (MFS) was introduced as a numerical method in the late seventies in a paper by Mathon and Johnston [18], followed shortly afterwards by applications to potential problems in papers by Fairweather and Johnston [6, 14]. Since then it has been applied to a wide range of problems in engineering science [2, 5, 7, 8, 21]. In the MFS, the approximate solution is taken as a linear superposition of singular solutions (fundamental solutions or Green's functions) of the differential operator for the problem of interest. As such, the approximate solution will satisfy the governing

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differential equation for the problem provided all singular points are located outside the domain of interest. In the traditional MFS approach, the coefficients in the MFS expansion are determined through either a linear or non-linear least-squares formulation using collocation. The linear approach utilizes source locations exterior to the domain of interest which are fixed *a priori*. In contrast, the nonlinear approach determines the source locations (also required to be exterior to the domain of interest) as part of the solution. Collocation points are selected on the physical boundary where the boundary conditions for the MFS approximations are minimized. It should be mentioned that the theoretical aspects of a Galerkin MFS for the solution of exterior Helmholtz problems were studied in [16] and a Galerkin-type MFS for harmonic problems was proposed in [9] but, to the best of our knowledge, there has been no follow-up since.

One problem with the strong approximate solution obtained by collocation methods is that one usually expects significant error to occur between the collocation points. One way to circumvent this issue is by employing a Galerkin approach where the boundary residuals are minimized in an average sense over the entire boundary instead of just at the collocation points. Galerkin approaches may also be employed in boundary element formulations (and other formulations) to also obtain symmetric coefficient matrices [17, 22].

The MFS has its origins in Trefftz's methods [15,23] which were originally developed as an alternative approach to Ritz's method for approximating the solution of partial differential equations. The primary difference between the two methods is that Trefftz's methods rely on the use of nonsingular basis functions which form a complete set of solutions to the differential equation, while the MFS utilizes singular fundamental solutions. It is worth noting that the approach of superposing singular solutions has been used for some time, see, for example, the application of singular source superposition described in [10] for problems in elastostatics. Trefftz methods can be developed from an indirect boundary integral equation [11] and the resulting minimization problem for the source strengths can be formulated for numerical computation either via a collocation approach or a Galerkin method [12]. A weighted residual approach leading to a Trefftz boundary element approach has also been discussed [19,20] where both collocation and Galerkin methods were used for the numerical solution of the problems considered. Interestingly, in [19] the authors found that both the collocation and Galerkin approaches yielded about the same degree of accuracy in the computations. It is also argued that the Galerkin approach is more economical since the resulting linear system is smaller than the one obtained with collocation. On the other hand, the Galerkin method requires numerical integration where collocation does not.

Our paper is organized as follows. After presenting the general formulation for the MFS in Section 2, we next detail the usual collocation formulation in Section 3 where implementation is also discussed. In sections 4 and 5 we present two alternative formulations for the Galerkin MFS and discuss implementation details. Symmetric coefficient matrices can be obtained from the Galerkin formulations presented and details concerning this symmetry are addressed in the Appendix. We finally discuss the Dirichlet prob-

lem for a circular domain in Section 6 with numerical results using both the collocation and Galerkin formulations given in Section 7.

2 The method of fundamental solutions

We consider Laplace's equation in \mathbb{R}^2

$$\Delta u = 0 \quad \text{in } \Omega, \tag{2.1a}$$

subject to Dirichlet boundary conditions

$$u = \bar{u} \quad \text{on } \partial\Omega_1, \tag{2.1b}$$

and Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \bar{q} \quad \text{on } \partial\Omega_2, \tag{2.1c}$$

where the boundary of the domain is $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ with $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$. In (2.1c), $\partial/\partial n$ denotes differentiation along the outward unit normal vector \mathbf{n} to the boundary $\partial\Omega_2$.

In the MFS we approximate the solution of Eq. (2.1a) by, see e.g., [7, 8],

$$u_N(P) = \sum_{j=1}^N c_j G(P, Q_j), \quad P \in \Omega, \tag{2.2}$$

where the $\{Q_j\}_{j=1}^N \in D$ are singularities located in $\mathbb{R}^2 \setminus \Omega$, which are assumed fixed and prescribed and $\{c_j\}_{j=1}^N$ are unknown coefficients to be determined. In (2.2), $G(P, Q)$ is a fundamental solution of the Laplace operator or Green's function, given by

$$G(P, Q) = -\frac{1}{2\pi} \log|P - Q|. \tag{2.3}$$

We next explore various ways of determining the coefficients $\{c_j\}_{j=1}^N$ in (2.2).

3 Collocation formulation

In the collocation formulation of the MFS, the value of $u_N(P)$ (or its normal derivative) is prescribed at the boundary points $\{P_i\}_{i=1}^M$ where the boundary value of u (or its normal derivative) is known. The resulting system of equations is linear in the case of the singularities Q_j are prescribed *a priori*, or nonlinear if the Q_j are also unknown [18]. In this study we shall apply the linear case, which leads to

$$u_N(P_i) = \bar{u}(P_i), \quad i = 1, \dots, M_1, \tag{3.1a}$$

$$\frac{\partial u_N}{\partial n}(P_i) = \bar{q}(P_i), \quad i = M_1 + 1, \dots, M. \tag{3.1b}$$

Upon solution the values of the coefficients $\{c_j\}_{j=1}^N$ appearing in (2.2) are known. Collocation techniques could lead to residual errors between the collocation points that may not be acceptable.

3.1 Implementation

System (3.1) may be re-written in the form

$$A^c c = b^c, \tag{3.2}$$

where $A^c \in \mathbb{R}^{M \times N}$ is defined by

$$A_{i,j}^c = \begin{cases} G(P_i, Q_j), & i=1, \dots, M_1, \\ \frac{\partial G}{\partial n}(P_i, Q_j), & i=M_1+1, \dots, M, \end{cases} \quad j=1, \dots, N, \tag{3.3}$$

$b \in \mathbb{R}^{M \times 1}$ is defined by

$$b_i^c = \begin{cases} \bar{u}(P_i), & i=1, \dots, M_1, \\ \bar{q}(P_i), & i=M_1+1, \dots, M, \end{cases} \tag{3.4}$$

and $c = [c_1, c_2, \dots, c_N]^T$. If $M > N$ system (3.2) is overdetermined and its solution is obtained using a linear least-squares solver whereas when $M = N$ the system is square and may be solved using Gaussian elimination. In this study we shall only consider the case $M = N$.

4 First Galerkin formulation

Alternatively, one could employ a Galerkin method where the boundary residual is minimized in an average sense over the entire boundary [22]. With reference to boundary value problem (2.1), we define the residual R as

$$R(P) = \begin{cases} u_N(P) - \bar{u}(P), & P \in \partial\Omega_1, \\ \frac{\partial u_N}{\partial n}(P) - \bar{r}(P), & P \in \partial\Omega_2. \end{cases} \tag{4.1}$$

In general, a boundary Galerkin statement of the general weighted residuals problem is

$$\int_{\partial\Omega} \Psi_k R ds = 0, \quad k=1, \dots, N, \tag{4.2}$$

where the Ψ_k are weight functions taken to be identical to the basis functions appearing in the chosen form of the approximate solution. In the case of the MFS, the residual we seek to minimize is

$$R(P) = \begin{cases} \sum_{j=1}^N c_j G(P, Q_j) - \bar{u}(P), & P \in \partial\Omega_1, \\ \sum_{j=1}^N c_j \frac{\partial G}{\partial n}(P, Q_j) - \bar{q}(P), & P \in \partial\Omega_2. \end{cases} \tag{4.3}$$

With reference to Eq. (2.1a), the weighting functions are taken as fundamental solutions, so the Galerkin statement becomes

$$\int_{\partial\Omega} G(P, Q_k) R(P) ds = 0, \quad k = 1, \dots, N, \tag{4.4}$$

or

$$\begin{aligned} & \int_{\partial\Omega_1} G(P, Q_k) \left(\sum_{j=1}^N c_j G(P, Q_j) - \bar{u}(P) \right) ds + \int_{\partial\Omega_2} G(P, Q_k) \left(\sum_{j=1}^N c_j \frac{\partial G}{\partial n}(P, Q_j) - \bar{q}(P) \right) ds \\ & = 0, \quad P \in \partial\Omega, \quad k = 1, \dots, N. \end{aligned} \tag{4.5}$$

With such a Galerkin approach we lose the integration-free advantage of the collocation formulation of the MFS, yet we gain with respect to the behavior of the boundary residual. We also note that the integrations needed to evaluate Eq. (4.4) are non-singular since the source point location $Q_j \notin \bar{\Omega}$, so standard quadrature methods should be sufficient. We also note that for fixed source locations Q_j , Eq. (4.4) yields N equations in the N unknown source strengths c_j .

4.1 Implementation

System (4.4) may be re-written in the form

$$A^{g_1} \mathbf{c} = \mathbf{b}^{g_1}, \tag{4.6}$$

where $A^{g_1} \in \mathbb{R}^{N \times N}$ is defined by

$$A_{k,j}^{g_1} = \int_{\partial\Omega_1} G(P, Q_k) G(P, Q_j) ds + \int_{\partial\Omega_2} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds, \quad k, j = 1, \dots, N, \tag{4.7}$$

and $\mathbf{b}^{g_1} \in \mathbb{R}^{N \times 1}$ is defined by

$$b_k^{g_1} = \int_{\partial\Omega_1} G(P, Q_k) \bar{u}(P) ds + \int_{\partial\Omega_2} G(P, Q_k) \bar{q}(P) ds, \quad k = 1, \dots, N. \tag{4.8}$$

In the case of the pure Dirichlet problem, that is when $\partial\Omega_2 = \emptyset$ in (2.1), it can be shown that the matrix A^{g_1} is symmetric and positive definite (see Theorem A.1 of Appendix) and hence the system (4.6) has a unique solution. In the evaluation of the elements in (4.7) and (5.5) we may write

$$A_{k,j}^{g_1} = \sum_{i=1}^{N_1} \int_{\partial\Omega_1^{(i)}} G(P, Q_k) G(P, Q_j) ds + \sum_{i=1}^{N_2} \int_{\partial\Omega_2^{(i)}} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds \tag{4.9a}$$

and

$$b_k^{g_1} = \sum_{i=1}^{N_1} \int_{\partial\Omega_1^{(i)}} G(P, Q_k) \bar{u}(P) ds + \sum_{i=1}^{N_2} \int_{\partial\Omega_2^{(i)}} G(P, Q_k) \bar{q}(P) ds, \quad k, j = 1, \dots, N, \tag{4.9b}$$

where $N = N_1 + N_2$,

$$\partial\Omega_\ell = \bigcup_{i=1}^{N_\ell} \partial\Omega_\ell^{(i)} \quad \text{and} \quad \partial\Omega_\ell^{(i)} \cap \partial\Omega_\ell^{(j)} = \emptyset \quad \text{if } i \neq j, \quad \text{for } \ell = 1, 2. \quad (4.10)$$

We shall assume that each of the $\{\partial\Omega_\ell^{(i)}\}_{i=1}^{N_\ell}$, $\ell = 1, 2$, is a straight line segment related to the boundary points $\{P_i\}_{i=1}^N$ by

$$\partial\Omega_\ell^{(i)} = \overline{P_i P_{i+1}}, \quad i = 1, \dots, N, \quad \text{with the convention that } P_{N+1} = P_1. \quad (4.11)$$

We next consider the evaluation of the integrals

$${}_1^i A_{k,j}^{g_1} = \int_{\partial\Omega_1^{(i)}} G(P, Q_k) G(P, Q_j) ds, \quad {}_1^i b_k^{g_1} = \int_{\partial\Omega_1^{(i)}} G(P, Q_k) \bar{u}(P) ds, \quad i = 1, \dots, N_1, \quad (4.12)$$

and

$${}_2^i A_{k,j}^{g_1} = \int_{\partial\Omega_2^{(i)}} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds, \quad {}_2^i b_k^{g_1} = \int_{\partial\Omega_2^{(i)}} G(P, Q_k) \bar{q}(P) ds, \quad i = 1, \dots, N_2, \quad (4.13)$$

for $k, j = 1, \dots, N$.

We first consider the evaluation of the integrals ${}_1^i A_{k,j}^{g_1}$, $\ell = 1, 2$. Let the coordinates of the end-points of the line segment $\partial\Omega_\ell^{(i)}$ be (x_1, y_1) and (x_2, y_2) , the coordinates of the source point Q_k be (x_{Q_k}, y_{Q_k}) and the coordinates of the source point Q_j be (x_{Q_j}, y_{Q_j}) . Then

$$\begin{aligned} {}_1^i A_{k,j}^{g_1} &= \int_{\partial\Omega_1^{(i)}} G(P, Q_k) G(P, Q_j) ds \\ &= \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}{8\pi^2} \int_{-1}^1 \log r_k(\xi) \log r_j(\xi) d\xi, \end{aligned} \quad (4.14a)$$

$$\begin{aligned} {}_2^i A_{k,j}^{g_1} &= \int_{\partial\Omega_1^{(i)}} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds \\ &= \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}{8\pi^2} \int_{-1}^1 \log r_k(\xi) \frac{(x_\xi - x_{Q_j})n_1 + (y_\xi - y_{Q_j})n_2}{r_j(\xi)^2} d\xi, \end{aligned} \quad (4.14b)$$

where

$$r_k(\xi) = \sqrt{(x_{Q_k} - x_\xi)^2 + (y_{Q_k} - y_\xi)^2}, \quad r_j(\xi) = \sqrt{(x_{Q_j} - x_\xi)^2 + (y_{Q_j} - y_\xi)^2},$$

and

$$x_\xi = \frac{(x_2 - x_1)\xi + (x_2 + x_1)}{2}, \quad y_\xi = \frac{(y_2 - y_1)\xi + (y_2 + y_1)}{2}.$$

Note that in (4.14a) we have used that

$$\frac{ds}{d\xi} = \sqrt{\left(\frac{dx_\xi}{d\xi}\right)^2 + \left(\frac{dy_\xi}{d\xi}\right)^2}.$$

Similarly, the integrals $b_k^{s_\ell^i}$, $\ell = 1, 2$ are evaluated by the formulæ

$$\begin{aligned} {}_1^i b_k^{s_1} &= \int_{\partial\Omega_1^{(i)}} G(P, Q_k) \bar{u}(P) ds \\ &= - \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}{4\pi} \int_{-1}^1 \log r_k(\xi) \bar{u}(x(\xi), y(\xi)) d\xi \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} {}_2^i b_k^{s_1} &= \int_{\partial\Omega_2^{(i)}} G(P, Q_k) \bar{q}(P) ds \\ &= - \frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}{4\pi} \int_{-1}^1 \log r_k(\xi) \bar{q}(x(\xi), y(\xi)) d\xi. \end{aligned} \tag{4.16}$$

Each of these integrals may be approximated by means of Gauss-Legendre quadrature, that is

$$\int_{-1}^1 F(\xi) d\xi = \sum_{l=1}^L w_l F(\xi_l), \tag{4.17}$$

where the ξ_l and w_l are the nodes and weights, respectively, of the L -point Gauss-Legendre quadrature which may be found in the literature (see e.g., [1]). In this study we shall be using the four-point Gauss-Legendre quadrature.

5 Second Galerkin formulation

Alternatively, instead of the Galerkin statement (4.4) one could employ the following statement [3]

$$\int_{\partial\Omega_1} \frac{\partial G}{\partial n}(P, Q_k) R(P) ds - \int_{\partial\Omega_2} G(P, Q_k) R(P) ds = 0, \quad k = 1, \dots, N, \tag{5.1}$$

or

$$\begin{aligned} &\int_{\partial\Omega_1} \frac{\partial G}{\partial n}(P, Q_k) \left(\sum_{j=1}^N c_j G(P, Q_j) - \bar{u}(P) \right) ds - \int_{\partial\Omega_2} G(P, Q_k) \left(\sum_{j=1}^N c_j \frac{\partial G}{\partial n}(P, Q_j) - \bar{q}(P) \right) ds \\ &= 0, \quad P \in \partial\Omega, \quad k = 1, \dots, N. \end{aligned} \tag{5.2}$$

The advantage of this formulation is the fact that it yields symmetric matrices for the mixed boundary conditions (2.1b)-(2.1b) as is demonstrated in the Appendix (Theorem A.2).

5.1 Implementation

System (5.2) may be re-written in the form

$$A^{g^2} \mathbf{c} = \mathbf{b}^{g^2}, \quad (5.3)$$

where $A^{g^2} \in \mathbb{R}^{N \times N}$ is defined by

$$A_{k,j}^{g^2} = \int_{\partial\Omega_1} \frac{\partial G}{\partial n}(P, Q_k) G(P, Q_j) ds - \int_{\partial\Omega_2} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds, \quad k, j = 1, \dots, N, \quad (5.4)$$

and $\mathbf{b}^{g^2} \in \mathbb{R}^{N \times 1}$ is defined by

$$b_k^{g^2} = \int_{\partial\Omega_1} \frac{\partial G}{\partial n}(P, Q_k) \bar{u}(P) ds - \int_{\partial\Omega_2} G(P, Q_k) \bar{q}(P) ds, \quad k = 1, \dots, N. \quad (5.5)$$

Dividing the boundary into segments as described by (4.10)-(4.11), we write

$$A_{k,j}^{g^2} = \sum_{i=1}^{N_1} \int_{\partial\Omega_1^{(i)}} \frac{\partial G}{\partial n}(P, Q_k) G(P, Q_j) ds - \sum_{i=1}^{N_2} \int_{\partial\Omega_2^{(i)}} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds, \quad (5.6a)$$

$$b_k^{g^2} = \sum_{i=1}^{N_1} \int_{\partial\Omega_1^{(i)}} \frac{\partial G}{\partial n}(P, Q_k) \bar{u}(P) ds - \sum_{i=1}^{N_2} \int_{\partial\Omega_2^{(i)}} G(P, Q_k) \bar{q}(P) ds, \quad k, j = 1, \dots, N, \quad (5.6b)$$

which means that we now need to evaluate the integrals

$${}^i_1 A_{k,j}^{g^2} = \int_{\partial\Omega_1^{(i)}} \frac{\partial G}{\partial n}(P, Q_k) G(P, Q_j) ds, \quad {}^i_1 b_k^{g^2} = \int_{\partial\Omega_1^{(i)}} \frac{\partial G}{\partial n}(P, Q_k) \bar{u}(P) ds, \quad i = 1, \dots, N_1, \quad (5.7)$$

and

$${}^i_2 A_{k,j}^{g^2} = \int_{\partial\Omega_2^{(i)}} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds, \quad {}^i_2 b_k^{g^2} = \int_{\partial\Omega_2^{(i)}} G(P, Q_k) \bar{q}(P) ds, \quad i = 1, \dots, N_2, \quad (5.8)$$

for $k, j = 1, \dots, N$. We note that the integrals (5.8) are the same as the corresponding integrals in (4.13). The integrals (5.7) are evaluated by using the Gauss-Legendre quadrature formula (4.17) as described in Section 4.1.

6 Dirichlet problems in circular domains

There are considerable simplifications and savings in the three proposed approaches when the domain Ω is a disk and we consider the Dirichlet problem (2.1a)-(2.1b). It is well-documented that in this case, the collocation approach yields coefficient matrices A^c in system (3.2) which are circulant [4,21]. In particular, if Ω is the disk $\Omega = \{\mathbf{x} \in \mathbb{R}^2: |\mathbf{x}| < \varrho\}$, we choose the collocation points $\{P_i\}_{i=1}^N$ such that

$$P_i = \varrho \left(\cos \frac{2(i-1)\pi}{N}, \sin \frac{2(i-1)\pi}{N} \right), \quad i = 1, \dots, N,$$

and the singularities $\{Q_j\}_{j=1}^N$ such that

$$Q_j = R \left(\cos \frac{2(j-1+\alpha)\pi}{N}, \sin \frac{2(j-1+\alpha)\pi}{N} \right), \quad j=1, \dots, N,$$

where $R > \varrho$ and the positions of the sources differ by an angle $2\pi\alpha/N$ from the positions of the boundary points and $0 \leq \alpha < 1$. In this case, the matrix A^c in (3.2) is circulant while the matrix A^{s_1} in (4.6) defined in the Dirichlet case by

$$A_{k,j}^{s_1} = \int_{\partial\Omega} G(P, Q_k) G(P, Q_j) ds \tag{6.1}$$

is both circulant and symmetric. In the second Galerkin approach we define the outward unit normal vector n_i for each segment $\partial\Omega^{(i)}$ by

$$n_i = \left(\cos \frac{(2i-1)\pi}{N}, \sin \frac{(2i-1)\pi}{N} \right), \quad i=1, \dots, N.$$

With this definition, the matrix A^{s_2} in (5.3) defined in the Dirichlet case by

$$A_{k,j}^{s_2} = \int_{\partial\Omega} \frac{\partial G}{\partial n}(P, Q_k) G(P, Q_j) ds \tag{6.2}$$

is circulant. A proof that the matrices defined by (6.1) and (6.2) are circulant is given in the Appendix (Theorem 3). Thus, in each of the three cases we have a system of the form

$$Ac = b, \tag{6.3}$$

where A is circulant. If we define the matrix $U \in \mathbb{C}^{N \times N}$ by

$$U = \frac{1}{\sqrt{N}} \left(e^{-2\pi i(k-1)(\ell-1)/N} \right)_{k,\ell=1}^N$$

we premultiply system (6.3) by U to obtain

$$UAU^*Uc = Ub \quad \text{or} \quad D\hat{c} = \hat{b},$$

where $\hat{b} = Ub$, $\hat{c} = Uc$ and the matrix $D = UAU^*$ is diagonal. The elements of $\hat{c} = [\hat{c}_1, \hat{c}_2, \dots, \hat{c}_N]$ can be easily calculated from $\hat{c}_i = \hat{b}_i / D_{ii}$, $i=1, \dots, N$ and then c can be recovered from $c = U^*\hat{c}$. Note that the operations $\hat{b} = Ub$, $D = UAU^*$ and $c = U^*\hat{c}$ are carried out efficiently using fast Fourier transforms (for details, see [21]).

7 Numerical results

7.1 Example 1

We first consider the solution of the Dirichlet boundary value problem (2.1a)-(2.1b) on the square $\Omega = (-1,1) \times (-1,1)$ using both the collocation and Galerkin formulations.

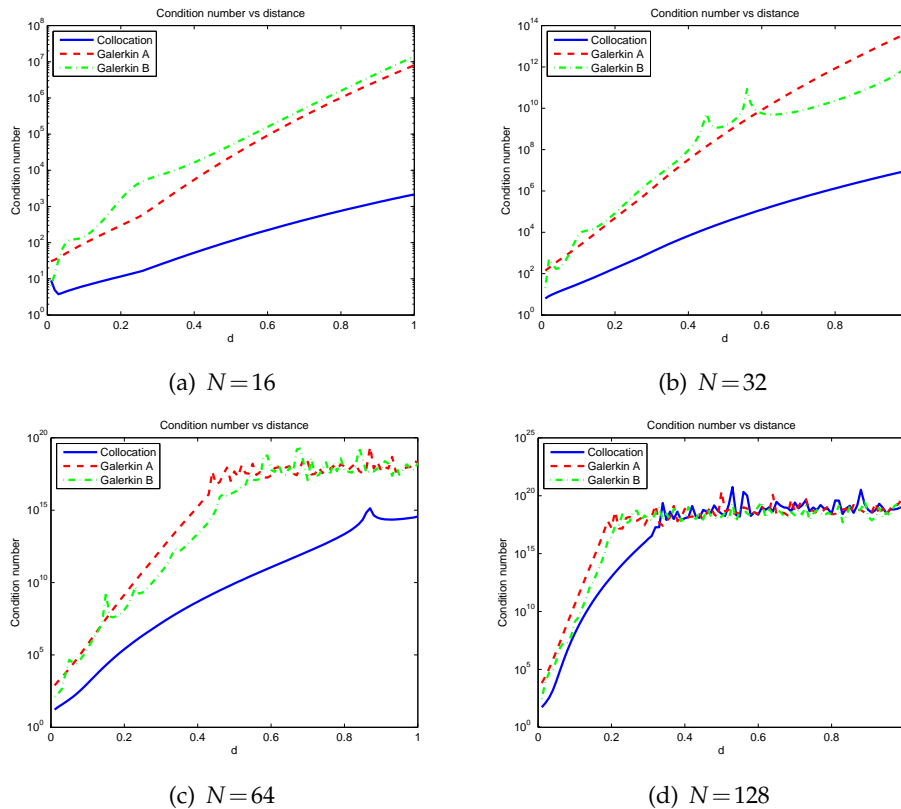


Figure 2: Example 1: Condition numbers versus distance of pseudo-boundary from boundary.

7.2 Example 2

We next consider the solution of the mixed boundary value problem (2.1a)-(2.1c) on the square $\Omega = (-1,1) \times (-1,1)$ using both the collocation and Galerkin formulations. As in Example 1, the boundary $\partial\Omega$ of Ω is subdivided into $N/4$ equal subintervals on each side of the square. The boundary segments $\partial\Omega_1$ and $\partial\Omega_2$ are defined as follows:

$$\partial\Omega_1 = \{(x,y) : -1 \leq x \leq 1, y = -1\}$$

and

$$\partial\Omega_2 = \{(x,y) : x = 1, -1 \leq y \leq 1\} \cup \{(x,y) : -1 \leq x \leq 1, y = 1\} \cup \{(x,y) : x = -1, -1 \leq y \leq 1\}.$$

The boundary conditions are taken to correspond to the exact solution $u(x,y) = e^x \cos y$. The maximum absolute error in the approximation u_N for the two formulations is calculated over 404 points on $\partial\Omega$ (101 equally distributed points on each side). In Fig. 3 we present the maximum absolute error for each formulation versus the distance d of the pseudo-boundary from the boundary for different numbers of degrees of freedom. From this figure we can observe that both Galerkin formulations perform in a similar

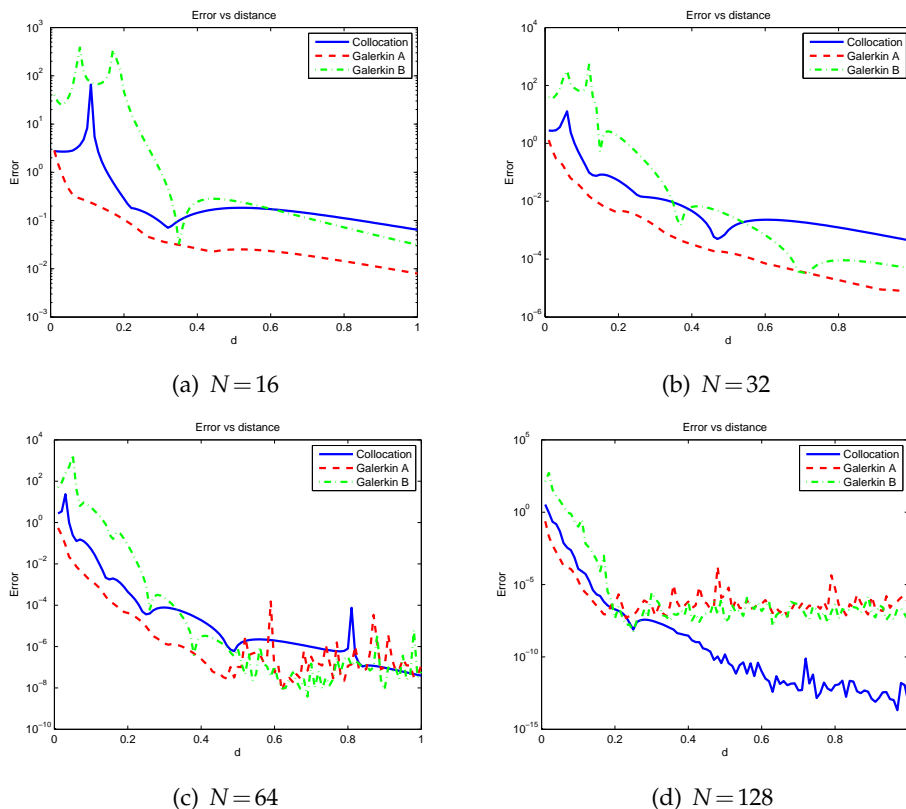


Figure 3: Example 2: Maximum absolute errors versus distance of pseudo-boundary from boundary.

way and both are inferior to collocation as N increases. In Fig. 4 we present plots of the 2-norm condition numbers of the matrices A^{g_1}, A^{g_2} and A^c versus the distance d for various numbers of degrees of freedom, where it can be observed that the behaviour of the condition numbers is similar to the corresponding condition numbers in Example 1.

7.3 Example 3

We next consider the solution of the Dirichlet boundary value problem (2.1a)-(2.1b) on the unit disk using both the collocation and Galerkin formulations. We use the efficient implementation described in Section 6 in both cases. The exact solution of the problem is $u(x, y) = e^x \cos y$ and the maximum absolute error in the approximation u_N for the three formulations is calculated over 401 uniformly distributed points on $\partial\Omega$. In Fig. 5 we present the maximum absolute error for each formulation versus the distance d of the pseudo-boundary from the boundary for different numbers of degrees of freedom. From this figure we observe that in this problem, the accuracy of collocation is considerably higher than that of the first Galerkin formulation. The second Galerkin formulation performs less well than the other two formulations. In Fig. 6 we present the maximum

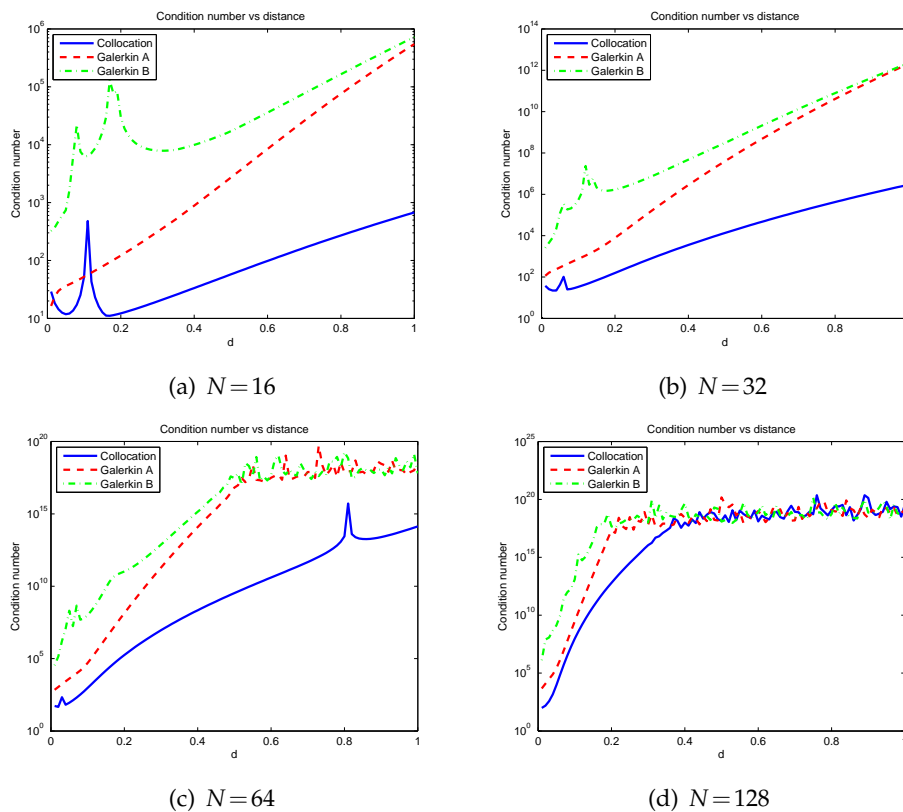


Figure 4: Example 2: Condition numbers versus distance of pseudo-boundary from boundary.

absolute errors versus the angular parameter α for $N=128$ for $R=1.05, 1.1$ and 1.15 . It appears that in the collocation approach the error is minimized as α approaches $1/4$, which is consistent with previous observations [21]. In contrast, the first Galerkin formulation appears to be unaffected by the rotation, while the second Galerkin formulation has its worst results as α approaches $1/4$.

8 Concluding remarks

We have presented two formulations of a Galerkin approach to solving Laplace problems with the method of fundamental solutions (MFS). In theory this approach should improve the distribution of error along the boundary when compared to the traditional collocation approach for the MFS. After formulating the Galerkin approach, we presented three test problems to illustrate the applicability of the method. In the first, a Dirichlet problem on a square was investigated, and for relatively few degrees of freedom the first Galerkin formulation yielded more accurate results than the collocation formulation. However, the second Galerkin formulation yielded poor results. It was noted that as the

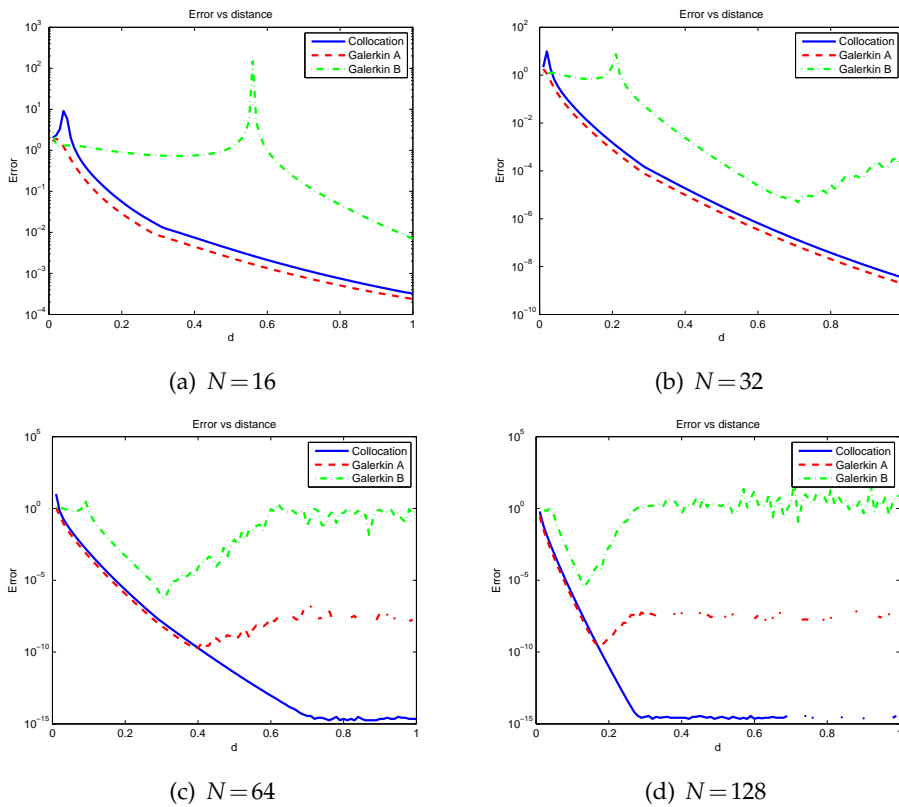


Figure 5: Example 3: Maximum absolute errors versus distance of pseudo-boundary from boundary.

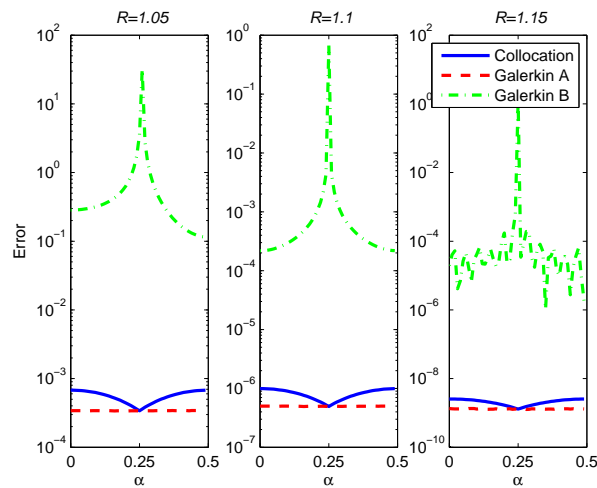


Figure 6: Example 3: Maximum absolute errors versus α .

number of degrees of freedom was increased, the accuracy of collocation approach became considerably higher than that of the first Galerkin formulation. This was explained

by the fact that the conditioning of the Galerkin coefficient matrices is much poorer than the conditioning of the corresponding collocation matrices. The second example problem investigated was a mixed boundary value problem on the square. We noted that both Galerkin formulations performed in a similar way, yet both were inferior to collocation as the number of degrees of freedom was increased. The final example considered was a Dirichlet problem on a disk, where it was observed that the accuracy of the collocation approach was considerably higher than that of the first Galerkin formulation, and the second Galerkin formulation performed less well than either collocation or the first Galerkin formulation. In light of these example problems, we conclude that, overall, the collocation approach can yield more accurate numerical approximations provided a sufficiently large number of degrees of freedom is used.

Appendix

Theorem A.1. *In the case of the pure Dirichlet boundary value problem (2.1a)-(2.1b), in the first Galerkin formulation, the matrix A^{s_1} in (4.6) is symmetric and positive definite.*

Proof. Clearly the matrix A^{s_1} in (4.7) is symmetric since $A_{k,j}^{s_1} = A_{j,k}^{s_1}$, $k, j = 1, \dots, N$. In addition, if

$$u_N(P) = \sum_{j=1}^N c_j G(P, Q_j),$$

then, following the arguments of, for example [13, Section 1.2],

$$\begin{aligned} c^T A^{s_1} c &= \sum_{k=1}^N \sum_{j=1}^N c_k c_j \int_{\partial\Omega} G(P, Q_k) G(P, Q_j) ds \\ &= \int_{\partial\Omega} \left\{ \sum_{k=1}^N \sum_{j=1}^N c_k c_j G(P, Q_k) G(P, Q_j) \right\} ds \\ &= \int_{\partial\Omega} \left\{ \left[\sum_{k=1}^N c_k G(P, Q_k) \right] \left[\sum_{j=1}^N c_j G(P, Q_j) \right] \right\} ds \\ &= \int_{\partial\Omega} [u_N(P)]^2 ds \geq 0. \end{aligned} \tag{A.1}$$

In (A.1) we have the equality only if $\int_{\partial\Omega} [u_N(P)]^2 ds = 0$, that is only if $u_N = 0$ on $\partial\Omega \iff u_N \equiv 0$ in $\Omega \iff c = \mathbf{0}$. We have thus shown that for $c \neq \mathbf{0}$, $c^T A^{s_1} c > 0$, which concludes the proof. □

Theorem A.2. *In the case of the mixed boundary value problem (2.1), in the second Galerkin formulation, the matrix A^{s_2} in (5.3) is symmetric.*

Proof. We have that the elements of the matrix A^{s_2} are given by (5.4). Following the corresponding formulation in [3], we know from Green’s identity that

$$\begin{aligned} & \int_{\Omega} G(P, Q_k) \Delta G(P, Q_j) d\omega - \int_{\Omega} G(P, Q_j) \Delta G(P, Q_k) d\omega \\ &= \int_{\partial\Omega} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds - \int_{\partial\Omega} G(P, Q_j) \frac{\partial G}{\partial n}(P, Q_k) ds, \quad k, j = 1, \dots, N. \end{aligned} \tag{A.2}$$

Moreover, since the functions $G(P, Q_k), k = 1, \dots, N$, are harmonic, (A.2) yields

$$\int_{\partial\Omega} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds - \int_{\partial\Omega} G(P, Q_j) \frac{\partial G}{\partial n}(P, Q_k) ds = 0 \tag{A.3}$$

or

$$\begin{aligned} & \int_{\partial\Omega_1} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds + \int_{\partial\Omega_2} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds \\ & - \int_{\partial\Omega_1} G(P, Q_j) \frac{\partial G}{\partial n}(P, Q_k) ds - \int_{\partial\Omega_2} G(P, Q_j) \frac{\partial G}{\partial n}(P, Q_k) ds = 0 \end{aligned} \tag{A.4}$$

or

$$\begin{aligned} & \int_{\partial\Omega_1} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds - \int_{\partial\Omega_2} G(P, Q_j) \frac{\partial G}{\partial n}(P, Q_k) ds \\ &= \int_{\partial\Omega_1} G(P, Q_j) \frac{\partial G}{\partial n}(P, Q_k) ds - \int_{\partial\Omega_2} G(P, Q_k) \frac{\partial G}{\partial n}(P, Q_j) ds, \end{aligned} \tag{A.5}$$

which from (5.4) gives that $A_{j,k}^{s_2} = B_{k,j}^s, k, j = 1, \dots, N$, hence the symmetry of the matrix A^{s_2} . The theorem is proved. \square

Theorem A.3. *The matrices defined by (6.1) and (6.2) corresponding to the two Galerkin formulations in Section 6 are circulant.*

Proof. In order to prove that a matrix $A \in \mathbb{R}^{N \times N}$ is circulant it suffices to show that

$$A_{kl} = A_{k+j, l+j}$$

and if $k+j > N$ (respectively $l+j > N$) then $k+j$ is replaced by $k+j-N$ (respectively by $l+j-N$).

Starting with matrix (6.1) we have that

$$A_{k,l}^{s_1} = \int_{\partial\Omega} G(P, Q_k) G(P, Q_l) ds = \frac{1}{4\pi^2} \int_0^{2\pi} \log|P-Q_k| \log|P-Q_l| \varrho d\vartheta. \tag{A.6}$$

By denoting the angles

$$\varphi_\ell = \frac{2(\ell-1+\alpha)\pi}{N}, \quad \ell = k, l, k+j, l+j,$$

we have that

$$\varphi_{\ell+j} = \varphi_\ell + \frac{2j\pi}{N}$$

and we can easily see from (A.6) that

$$A_{k,l}^{\delta_1} = \frac{1}{16\pi^2} \int_0^{2\pi} \log [q^2 + R^2 - 2qR\cos(\vartheta - \varphi_k)] [q^2 + R^2 - 2qR\cos(\vartheta - \varphi_l)] qd\vartheta. \quad (\text{A.7})$$

Similarly,

$$\begin{aligned} A_{k+j,l+j}^{\delta_1} &= \frac{1}{16\pi^2} \int_0^{2\pi} \log [q^2 + R^2 - 2qR\cos(\vartheta - \varphi_{k+j})] [q^2 + R^2 - 2qR\cos(\vartheta - \varphi_{l+j})] qd\vartheta \\ &= \frac{1}{16\pi^2} \int_0^{2\pi} \log \left[q^2 + R^2 - 2qR\cos\left(\vartheta - \varphi_k - \frac{2j\pi}{N}\right) \right] \left[q^2 + R^2 - 2qR\cos\left(\vartheta - \varphi_l - \frac{2j\pi}{N}\right) \right] qd\vartheta \\ &\quad \left(\text{with } \tilde{\vartheta} = \vartheta - \frac{2j\pi}{N}\right) \\ &= \frac{1}{16\pi^2} \int_{-\frac{2j\pi}{N}}^{2\pi - \frac{2j\pi}{N}} \log [q^2 + R^2 - 2qR\cos(\tilde{\vartheta} - \varphi_k)] [q^2 + R^2 - 2qR\cos(\tilde{\vartheta} - \varphi_l)] qd\tilde{\vartheta} \\ &= \frac{1}{16\pi^2} \int_{-\frac{2j\pi}{N}}^0 \log [q^2 + R^2 - 2qR\cos(\tilde{\vartheta} - \varphi_k)] [q^2 + R^2 - 2qR\cos(\tilde{\vartheta} - \varphi_l)] qd\tilde{\vartheta} + A_{k,l}^{\delta_1} \\ &\quad - \frac{1}{16\pi^2} \int_{2\pi - \frac{2j\pi}{N}}^{2\pi} \log [q^2 + R^2 - 2qR\cos(\tilde{\vartheta} - \varphi_k)] [q^2 + R^2 - 2qR\cos(\tilde{\vartheta} - \varphi_l)] qd\tilde{\vartheta}. \quad (\text{A.8}) \end{aligned}$$

Due to the 2π -periodicity of $\cos \tilde{\vartheta}$ and $\sin \tilde{\vartheta}$, the first and last terms in (A.8) cancel out yielding the desired result.

For matrix (6.2) we have that

$$\begin{aligned} A_{k,l}^{\delta_2} &= \int_{\partial\Omega} \frac{\partial G}{\partial n}(P, Q_k) G(P, Q_l) ds \\ &= \frac{1}{8\pi^2} \int_0^{2\pi} \log [q^2 + R^2 - 2qR\cos(\vartheta - \varphi_l)] \left[\frac{(q\cos\vartheta - R\cos\varphi_k)\cos\vartheta + (q\cos\vartheta - R\sin\varphi_k)\sin\vartheta}{q^2 + R^2 - 2qR\cos(\vartheta - \varphi_k)} \right] qd\vartheta \\ &= \frac{1}{8\pi^2} \int_0^{2\pi} \log [q^2 + R^2 - 2qR\cos(\vartheta - \varphi_l)] \frac{(q - R\cos(\vartheta - \varphi_k))}{q^2 + R^2 - 2qR\cos(\vartheta - \varphi_k)} qd\vartheta. \end{aligned}$$

Similarly, we have that

$$A_{k+j,l+j}^{\delta_2} = \frac{1}{8\pi^2} \int_0^{2\pi} \log \left[q^2 + R^2 - 2qR\cos\left(\vartheta - \varphi_l - \frac{2j\pi}{N}\right) \right] \frac{\left(q - R\cos\left(\vartheta - \varphi_k - \frac{2j\pi}{N}\right) \right)}{q^2 + R^2 - 2qR\cos\left(\vartheta - \varphi_k - \frac{2j\pi}{N}\right)} qd\vartheta$$

(and proceeding exactly as in the previous case)

$$= \frac{1}{8\pi^2} \int_{-\frac{2j\pi}{N}}^{2\pi - \frac{2j\pi}{N}} \log [q^2 + R^2 - 2qR\cos(\tilde{\vartheta} - \varphi_l)] \frac{(q - R\cos(\tilde{\vartheta} - \varphi_k))}{q^2 + R^2 - 2qR\cos(\tilde{\vartheta} - \varphi_k)} qd\tilde{\vartheta} = A_{k,l}^{\delta_2}. \quad (\text{A.9})$$

This completes the proof of the theorem. □

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