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Cell Conservative Flux Recovery and A Posteriori Error Estimate of Vertex-Centered Finite Volume Methods

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Abstract. A cell conservative flux recovery technique is developed here for vertex-centered finite volume methods of second order elliptic equations. It is based on solving a local Neumann problem on each control volume using mixed finite element methods. The recovered flux is used to construct a constant free *a posteriori* error estimator which is proven to be reliable and efficient. Some numerical tests are presented to confirm the theoretical results. Our method works for general order finite volume methods and the recovery-based and residual-based *a posteriori* error estimators is the first result on *a posteriori* error estimators for high order finite volume methods.

AMS subject classifications: 65N15, 65N30, 65N50

Key words: Finite volume methods, flux recovery, a posteriori error estimates.

1 Introduction

In this paper, we consider vertex-centered finite volume methods for solving diffusion type elliptic equation

$$-\nabla\cdot(\boldsymbol{K}\nabla u)=f \quad \text{in } \Omega, \tag{1.1}$$

with suitable Dirichlet or Neumann boundary conditions. Here $\Omega\subset\mathbb{R}^d$ is a polyhedral domain ($d\geq 2$), the diffusion coefficient $\boldsymbol{K}(\boldsymbol{x})$ is a $d\times d$ symmetric matrix function that is uniformly positive definite on Ω with components in $L^\infty(\Omega)$, and $f\in L^2(\Omega)$. An obvious virtue of finite volume method (FVM) is the local conservation property, which can be fundamental for the simulation of many physical models, e.g., in dynamics of of fluids in porous media.

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We shall recover a cell-conservative flux from vertex-centered FVMs and use the recovered flux to construct a constant free *a posteriori* error estimator, which will be called recovery-based error estimator. We obtain the reliability, i.e., upper bound, of the recovery-based error estimator through the so-called hypercircle method established by Prager and Synge [32]. We establish the efficiency, i.e., the local lower bound, of the recovery-based error estimator by showing that the recovery-based error estimator is locally equivalent to the well-known residual-based *a posteriori* error estimator. As a by product, we get the reliability of the residual-based error estimator for high order FVMs, which seems difficult to obtain using the same approach as finite element methods (FEMs) due to the loss of the Galerkin orthogonality.

To facilitate the discussion of the results, let us briefly introduce FVM. Many physical models can be written as the following balance equation:

$$-\int_{\partial b}(\boldsymbol{K}\nabla u)\cdot\boldsymbol{n}dS=\int_{b}fd\boldsymbol{x}\quad\text{for all}\ \ b\subset\Omega. \tag{1.2}$$

The discretization of (1.2) by choosing an appropriate finite element space $\mathbb{V}$ to approximate u and a finite number of subdomains b, the so-called control volume, will be called FVMs. There are mainly two types of FVMs, different in the choice of control volumes. Given a grid $\mathcal{T}$ of Ω, if we choose the cell of $\mathcal{T}$ as the control volume and associate the unknown to cells, we obtain cell-centered FVMs [10,33]; if we construct control volume for each vertex and associate the unknown to vertex, we obtain vertex-centered FVMs [7,14,21]. We shall consider vertex-centered FVMs in this paper.

Let u_h be a vertex-centered FVM approximation of Eq. (1.1). By solving a local problem with $\boldsymbol{K}\nabla u_h\cdot\boldsymbol{n}$ as Neumann boundary condition on each control volume with mixed finite element methods, we are able to recover a cell conservative flux from u_h. The local problem is well defined since the compatible condition

$$-\int_{\partial b}(\boldsymbol{K}\nabla u_h)\cdot\boldsymbol{n}dS=\int_{b}fd\boldsymbol{x},$$

is built into the vertex-centered FVM. The approximated flux $\boldsymbol{\sigma}_h$ will be sought in Raviart-Thomas (RT) or Brezzi-Douglas-Marini (BDM) spaces, depending on the order of the approximation u_h. By solving local problems in all control volumes, we obtain a cell-conservative flux on the original grid $\mathcal{T}$ and also a refinement of $\mathcal{T}$. Flux recovery techniques based on the solution of FEM have been extensively studied by many researchers; see [11] for $H(\mathrm{div})$-conforming flux approximation, and [15,16] for cell-conservative flux approximation. Flux recovery techniques based on the solution of linear FVM can be found in [42]. The new proposed flux recovery method based on the solution of FVM seems natural, much simpler and works also for high order FVM methods.

We prove that the recovered cell-conservative flux is of the same order of approximation as the finite volume approximation in the energy norm. Therefore this local recovery procedure provides an efficient way to compute a cell conservative flux for Eq. (1.1). Note that the quadratic finite volume method proposed in [14] on rectangular grids results in