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Solutions of Fractional Partial Differential Equations of Quantum Mechanics

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Abstract. The aim of this article is to investigate the solutions of generalized fractional partial differential equations involving Hilfer time fractional derivative and the space fractional generalized Laplace operators, occurring in quantum mechanics. The solutions of these equations are obtained by employing the joint Laplace and Fourier transforms, in terms of the Fox's H-function. Several special cases as solutions of one dimensional non-homogeneous fractional equations occurring in the quantum mechanics are presented. The results given earlier by Saxena et al. [Fract. Calc. Appl. Anal., 13(2) (2010), pp. 177–190] and Purohit and Kalla [J. Phys. A Math. Theor., 44 (4) (2011), 045202] follow as special cases of our findings.

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1 Introduction

The partial differential equations of fractional order have been successfully used for modeling some relevant physical processes, therefore, a large amount of research in the solutions of these equations has been published in the literature. Debnath [3] has discussed solutions of the various type of partial differential equations occurring in the fluid mechanics. Nikolova and Boyadjiev [17] found solution of the time-space fractional diffusion equations by means of the fractional generalization of Fourier transform and the classical Laplace transform. Solution of generalized diffusion equation containing two space-fractional derivatives have been recently analyzed by Pagnini and Mainardi [19]. Solutions of fractional reaction-diffusion equations are investigated in a number of recent papers by Saxena et al. [23–25] and Haubold et al. [6].

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Laskin [9–11] constructed space fractional quantum mechanics and formulated the fractional Schrödinger equation by generalizing the Feynman path integrals from Brownian-like to Lévy-like quantum mechanical paths. The Schrödinger equation thus obtained containing the space and time fractional derivatives. Several authors including Naber [16] and Saxena et al. [26,27] studied various aspects of the Schrödinger equations in terms of the fractional derivatives as dimensionless objects. For some physical applications of fractional Schrödinger equation, one can refer the work of Guo and Xu [5].

The Grunwald-Letnikov method is employed by Scherer et al. [28, 29] to solve fractional differential equations numerically.

In a recent paper, Purohit and Kalla [21] have investigated solutions of generalized fractional partial differential equations by employing the joint Laplace and Fourier transforms. Several special cases in terms of the solutions of one dimensional non-homogeneous fractional equations occurring in the fluid and quantum mechanics (diffusion, wave and Schrödinger equations) are also presented in the same paper.

The object of this paper is to investigate solutions of generalized fractional partial differential equations involving Hilfer time-fractional derivative and the space-fractional generalized Laplace operators by employing the joint Laplace and Fourier transforms. Several special cases in terms of the solutions of one dimensional non-homogeneous fractional equations occurring in the quantum mechanics are presented. It is to be noted that the problem considered here (involving Hilfer time-fractional derivative) is different than those considered by Saxena et al. [26] and the authors [21], where Caputo time-fractional derivative and Liouville space-fractional derivative were employed. Additionally, the problem considered here is more general than the problem considered by Saxena et al. [27]. Hilfer fractional derivative has advantage that it generalizes the Riemann-Liouville and Caputo type fractional derivative operators, therefore, several authors called this a general operator. The results given earlier by Saxena et al. [27] and Purohit and Kalla [21] follow as special cases of our findings.

In order to obtain the solutions of generalized fractional partial differential equations, the definitions and notations of the well-known operators are described below:

The Laplace transform of a function $U=U(x,t)$ (which is supposed to be continuous or sectionally continuous, and of exponential order as $t\to\infty$) with respect to the variable t is defined by

$$\mathcal{L}\{U(x,t)\}=\widehat{U}(x,s)=\int_0^\infty e^{-st}U(x,t)dt, \quad (t>0, \quad x\in\mathbb{R}), \tag{1.1}$$

where $\Re(s)>0$, and the inverse Laplace transform of $\widehat{U}(x,s)$ with respect to s is given by

$$\mathcal{L}^{-1}\{\widehat{U}(x,s)\}=U(x,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}\widehat{U}(x,s)ds, \tag{1.2}$$

where γ being a fixed real quantity.

The Fourier transform of a function $U(x,t)$ with respect to x is defined as:

$$\mathcal{F}\{U(x,t)\}=U^*(k,t)=\int_{-\infty}^{\infty}e^{ikx}U(x,t)dx, \quad (-\infty<k<\infty). \tag{1.3}$$

The inverse Fourier transform of a function $U^*(k,t)$ with respect to k is given by

$$\mathcal{F}^{-1}\{U^*(k,t)\}=U(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}U^*(k,t)dk. \tag{1.4}$$

The well-known Riemann-Liouville fractional integral operator of a function $U(x,t)$ of arbitrary order μ (cf. [15], pp. 45) is defined by

$${}_0D_t^{-\mu}U(x,t)=\frac{1}{\Gamma(\mu)}\int_0^t(t-u)^{\mu-1}U(x,u)du, \quad (\Re(\mu)>0). \tag{1.5}$$

The fractional derivative of arbitrary order μ is given by (see [22], pp. 37):

$${}_0D_t^{\mu}U(x,t)=\frac{1}{\Gamma(n-\mu)}\frac{\partial^n}{\partial t^n}\int_0^t(t-u)^{n-\mu-1}U(x,u)du, \quad (t>0, \quad n=[\mu]+1), \tag{1.6}$$

where $[\mu]$ represents the integral part of the number μ.

Like the Laplace transform of integer order derivative, it is easy to show that the Laplace transform of fractional derivative is given by (for details, see [18], pp. 134)

$$\mathcal{L}\{{}_0D_t^{\mu}U(x,t);s\}=s^{\mu}\widehat{U}(x,s)-\sum_{r=1}^{n}c_r s^{r-1}, \quad (n-1<\mu<n), \tag{1.7}$$

where

$$c_r=[{}_0D_t^{\mu-r}U(x,t)]_{t=0} \tag{1.8}$$

represents the initial conditions. Due to lack of physical interpretation of these initial conditions, Caputo [2] introduced the following definition of fractional derivative

$${}_0^C D_t^{\alpha}U(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-u)^{n-\alpha-1}U^{(n)}(x,u)du, \quad (t>0), \tag{1.9}$$

where $n-1<\alpha<n$, $n\in\mathbb{Z}$ and $U^{(n)}(x,t)$ is the partial derivative of order n of the function $U(x,t)$ with respect to the variable t. The initial conditions in Riemann-Liouville and Caputo type fractional derivatives are analyzed in details by Mainardi and Gorenflo [12].

The Laplace transform of Caputo's fractional derivative is given by (see [2,20]):

$$\mathcal{L}\{{}_0^C D_t^{\alpha}U(x,t);s\}=s^{\alpha}\widehat{U}(x,s)-\sum_{r=0}^{n-1}s^{\alpha-r-1}U^{(r)}(x,0), \quad (n-1<\alpha\le n). \tag{1.10}$$

The above formula play an important role in deriving the solution of differential and integral equations of fractional order governing certain physical problems of reaction and diffusion. One may refer to the monographs by Podlubny [20], Samko et al. [22], Mathai et al. [13] and Kilbas et al. [8] and recent papers [3, 14, 16, 21] and [23–27] on the subject.

A generalization of the Riemann-Liouville fractional derivative operator (1.6) and Caputo fractional derivative operator (1.9) is given by Hilfer [7], by introducing a fractional derivative operator of two parameters of order $0<\mu<1$ and type $0\le\nu\le1$ in the form

$$ {}_0D^{\mu,\nu}_{a+}U(x,t)=I^{\nu(1-\mu)}_{a+}\frac{\partial}{\partial t}(I^{(1-\nu)(1-\mu)}_{a+}U(x,t)). \tag{1.11}$$

It is to be observed that for $\nu=0$, (1.11) reduces to the classical Riemann-Liouville fractional derivative operator (1.6). On the other hand, for ν=1 it yields the Caputo fractional derivative operator defined by (1.9). In the same paper Hilfer [7], given the Laplace transformation formula for this operator as under:

$$L\{{}_0D^{\mu,\nu}_{0+}U(x,t);s\}=s^{\mu}\widehat{U}(x,s)-s^{\nu(\mu-1)}I^{(1-\nu)(1-\mu)}_{0+}U(x,0+),\quad (0<\mu<1), \tag{1.12}$$

where the initial value term

$$I^{(1-\nu)(1-\mu)}_{0+}U(x,0+),$$

involves the Riemann-Liouville fractional integral operator of order $(1-\nu)(1-\mu)$ evaluated in the limit as $t\to0+$ and $\widehat{U}(x,s)$ is the Laplace transform of the function $U(x,t)$.

Following Brockmann and Sokolev [1], a symmetric fractional generalization of the Laplace operator is defined as

$$\Delta^{\alpha/2}=\frac{1}{2\cos(\pi\alpha/2)}\{{}_{-\infty}D^{\alpha}_{x}+{}_{x}D^{\alpha}_{\infty}\},\quad (0<\alpha\le2), \tag{1.13}$$

where the operators on the right of (1.13) are defined by

$${}_{-\infty}D^{\alpha}_{x}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{x}\frac{f^{(n)}(u)}{(x-u)^{\alpha+1-n}}du,\quad (n=[\alpha]+1), \tag{1.14}$$

and

$${}_{x}D^{\alpha}_{\infty}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_{x}^{\infty}\frac{f^{(n)}(u)}{(u-x)^{\alpha+1-n}}du,\quad (n=[\alpha]+1). \tag{1.15}$$

A key role in our discussion is given to a relation established in [1], according to which

$$\mathcal{F}\{\Delta^{\alpha/2}U(x,t);k\}=-|k|^{\alpha}U^{*}(k,t),\quad (0<\alpha\le2), \tag{1.16}$$

where $U^{*}(k,t)$ is the Fourier transform of $U(x,t)$.

The H-function is defined by means of a Mellin-Barnes type integral in the following manner (see [13]):

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z \Big| \begin{array}{c} (a_p,A_p) \\ (b_q,B_q) \end{array} \right] \\ &= H_{p,q}^{m,n}\left[z \Big| \begin{array}{c} (a_1,A_1),\cdots,(a_p,A_p) \\ (b_1,B_1),\cdots,(b_q,B_q) \end{array} \right] = \frac{1}{2\pi i}\int_L \Theta(\xi) z^{-\xi} d\xi, \end{aligned} \tag{1.17}$$

where $i=(-1)^{1/2}$,

$$\Theta(\xi) = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod\limits_{i=1}^{n}\Gamma(1-a_i-A_i\xi)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_j\xi)\prod\limits_{j=n+1}^{p}\Gamma(a_i+A_i\xi)}, \tag{1.18}$$

and an empty product is always interpreted as unity; $m,n,p,q \in \mathbb{N}_0$ with $0\le n\le p$, $1\le m\le q$, $A_i,B_j\in\mathbb{R}_+$, $a_i,b_j\in\mathbb{R}$ or $\mathbb{C}$ $(i=1,\cdots,p;\ j=1,\cdots,q)$, such that

$$A_i(b_j+k)\neq B_j(a_i-\ell-1), \quad (k,\ell\in N_0; \quad i=1,\cdots,n; \quad j=1,\cdots,m), \tag{1.19}$$

where we employ the usual notations: $\mathbb{N}_0=(0,1,2\cdots,)$; $\mathbb{R}=(-\infty,\infty)$, $\mathbb{R}_+=(0,\infty)$ and $\mathbb{C}$ being the complex number field.

2 Solutions of generalized fractional partial differential equations

In this section, we investigate the solution of certain generalized fractional partial differential equations of quantum mechanics.

Theorem 2.1. *Consider the following one dimensional non-homogeneous generalized fractional partial differential equation*

$${}_0D_t^{\mu,\nu}U(x,t) = a\Delta^{\alpha/2}U(x,t) + b\Delta^{\beta/2}V(x,t), \quad (x\in\mathbb{R}, \quad t>0, \quad 0<\alpha\le 2, \quad 0<\beta\le 2), \tag{2.1}$$

with initial conditions

$$I_{0+}^{(1-\nu)(1-\mu)}U(x,0+) = U_0(x), \quad (-\infty<x<\infty, \quad 0<\mu<1, \quad 0\le\nu\le 1), \tag{2.2}$$

and

$$U(x,t)\to 0 \ \text{ as } \ |x|\to\infty, \tag{2.3}$$

where the generalized fractional derivative ${}_0D_t^{\mu,\nu}$ *with respect to* t *is given by* (1.11) *and the operator* $\Delta^{\alpha/2}$ *is the fractional generalization of the Laplace operator, defined by* (1.13). *a and b*

are arbitrary constants, $U_0(x)$ and $V(x,t)$ are given functions. Then the solution of (2.1) *under the given conditions, is given by*

$$\begin{aligned} U(x,t)=&\int_{-\infty}^{\infty} G_1(x-\xi,t)U_0(\xi)d\xi \\ &-b\int_0^t (t-\tau)^{\mu-1}\Big[\int_{-\infty}^{\infty} G_2(x-\xi,t-\tau)V(\xi,\tau)d\xi\Big]d\tau, \end{aligned} \tag{2.4}$$

where the Green functions $G_1(x,t)$ and $G_2(x,t)$ are given by

$$G_1(x,t)=\frac{t^{\mu+\nu(1-\mu)-1}}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{a^{1/\alpha}t^{\mu/\alpha}}\left| \begin{matrix} (1,1/\alpha),(\mu+\nu(1-\mu),\mu/\alpha),\Big(1,\frac{1}{2}\Big) \\ (1,1/\alpha),(1,1),\Big(1,\frac{1}{2}\Big) \end{matrix}\right.\right], \tag{2.5}$$

and

$$G_2(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}|k|^{\beta}E_{\mu,\mu}(-a|k|^{\alpha}t^{\mu})dk, \tag{2.6}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function [4] *defined by*

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)}, \quad (\alpha,\beta\in\mathbb{C};\quad \Re(\alpha)>0,\quad \Re(\beta)>0). \tag{2.7}$$

Proof. In order to prove the theorem, we introduce the joint Laplace-Fourier transform (the Laplace transform with respect to variable t and Fourier transform with respect to variable x) in the form

$$\widehat{U}^*(k,s)=\int_0^{\infty}\int_{-\infty}^{\infty} e^{-st+ikx}U(x,t)dxdt, \quad (\Re(s)>0,\quad k\in\mathbb{R}). \tag{2.8}$$

On applying the operator (2.8) to the Eq. (2.1), under the valid conditions (2.2) and (2.3) and making use of the relations (1.12) and (1.16), we obtain

$$s^{\mu}\,\widehat{U}^*(k,s)-s^{\nu(\mu-1)}U_0^*(k)=-a|k|^{\alpha}\,\widehat{U}^*(k,s)-b|k|^{\beta}\,\widehat{V}^*(k,s), \tag{2.9}$$

where the symbols "$\widehat{}$" and " $*$ " indicates, respectively, the Laplace transform and Fourier transforms. Also k is the Fourier transform variable and s is the Laplace transform variable. Hence, the above Eq. (2.9), gives the following transform solution

$$\widehat{U}^*(k,s)=\frac{s^{\nu(\mu-1)}U_0^*(k)}{s^{\mu}+a|k|^{\alpha}}-\frac{b|k|^{\beta}\,\widehat{V}^*(k,s)}{s^{\mu}+a|k|^{\alpha}}. \tag{2.10}$$

To recover the original function $U(x,t)$ from (2.10), it is convenient first to invert the Laplace transform and then the Fourier transform.

The inverse Laplace transform leads to the following result

$$U^*(k,t)=\mathcal{L}^{-1}\Big[\frac{s^{\nu(\mu-1)}U_0^*(k)}{s^\mu+a|k|^\alpha}\Big]-b\mathcal{L}^{-1}\Big[\frac{|k|^\beta\,\widehat{V}^*(k,s)}{s^\mu+a|k|^\alpha}\Big]. \tag{2.11}$$

Thus, on using the well-known formula

$$\mathcal{L}^{-1}\Big[\frac{s^{\alpha-1}}{a+s^\beta};t\Big]=t^{\beta-\alpha}E_{\beta,\beta-\alpha+1}(-at^\beta), \tag{2.12}$$

and the convolution theorem for Laplace transform, we obtain

$$\begin{aligned}U^*(k,t)=&U_0^*(k)t^{\mu+\nu(1-\mu)-1}E_{\mu,\mu+\nu(1-\mu)}(-a|k|^\alpha t^\mu)\\&-b|k|^\beta\int_0^t(t-\tau)^{\mu-1}E_{\mu,\mu}(-a|k|^\alpha(t-\tau)^\mu)V^*(k,\tau)d\tau.\end{aligned} \tag{2.13}$$

Finally, the inverse Fourier transform gives the exact solution for the Eq. (2.1) as follows:

$$\begin{aligned}U(k,t)=&t^{\mu+\nu(1-\mu)-1}\mathcal{F}^{-1}\big[U_0^*(k)E_{\mu,\mu+\nu(1-\mu)}(-a|k|^\alpha t^\mu)\big]\\&-b\int_0^t(t-\tau)^{\mu-1}\mathcal{F}^{-1}\big[|k|^\beta E_{\mu,\mu}(-a|k|^\alpha(t-\tau)^\mu)V^*(k,\tau)\big]d\tau.\end{aligned} \tag{2.14}$$

If we now apply the convolution theorem of the Fourier transform to (2.14) and make use of the following inverse Fourier transform formula given by Haubold et al. [6], namely

$$\mathcal{F}^{-1}[E_{\beta,\gamma}(-at^\beta|k|^\alpha);x]=\frac{1}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{a^{1/\alpha}t^{\beta/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(\gamma,\beta/\alpha),\left(1,\frac{1}{2}\right)\\(1,1/\alpha),(1,1),\left(1,\frac{1}{2}\right)\end{matrix}\right], \tag{2.15}$$

where $\min\{\Re(\alpha),\Re(\beta),\Re(\gamma)\}>0$ and $\alpha>0$, it gives the solution in the form of (2.4). ☐

Now, we consider another variation of the above theorem. Let us replace the function $V(x,t)$ by the unknown function $U(x,t)$, we obtain the following:

Theorem 2.2. *Consider the following one dimensional fractional differential equation*

$${}_0D_t^{\mu,\nu}U(x,t)=a\Delta^{\alpha/2}U(x,t)+b\Delta^{\beta/2}U(x,t),\quad (x\in\mathbb{R},\ t>0,\ 0<\alpha\le 2,\ 0<\beta\le 2), \tag{2.16}$$

with initial conditions

$$I_{0+}^{(1-\nu)(1-\mu)}U(x,0+)=U_0(x),\quad (-\infty<x<\infty,\quad 0<\mu<1,\quad 0\le\nu\le 1), \tag{2.17}$$

and

$$U(x,t)\to 0\ \text{ as }\ |x|\to\infty, \tag{2.18}$$

where a and b are arbitrary constants, and $U_0(x)$ is a given function, then for solution of (2.16) *under the given conditions, there holds the formula*

$$U(x,t)=\int_{-\infty}^{\infty} G_3(x-\xi,t)U_0(\xi)d\xi, \tag{2.19}$$

where the Green functions $G_3(x,t)$ is given by

$$G_3(x,t)=\frac{t^{\mu+\nu(1-\mu)-1}}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\mu,\mu+\nu(1-\mu)}[-(a|k|^{\alpha}+b|k|^{\beta})t^{\mu}]dk. \tag{2.20}$$

3 Special cases

In this section, we consider some consequences and applications of the main results derived in the preceding section.

(a) If we set $a=i\hbar/(2m)$ and $\beta=0$, Theorem 2.1 yields to the solution of non-homogeneous fractional generalized Schrödinger equation as follows:

Corollary 3.1. *Consider the following one dimensional non-homogeneous generalized fractional Schrödinger equation of a particle of mass m, defined by*

$${}_0D_t^{\mu,\nu}U(x,t)=(i\hbar/2m)\Delta^{\alpha/2}U(x,t)+bV(x,t), \quad (x\in\mathbb{R}, \quad t>0, \quad 0<\alpha\le 2), \tag{3.1}$$

with initial conditions

$$I_{0+}^{(1-\nu)(1-\mu)}U(x,0+)=U_0(x), \quad (-\infty<x<\infty, \quad 0<\mu<1, \quad 0\le\nu\le 1), \tag{3.2}$$

and

$$U(x,t)\to 0 \ \text{ as } \ |x|\to\infty, \tag{3.3}$$

where b is arbitrary, $h=2\pi\hbar$ is the Plank constant, and $U_0(x)$ and $V(x,t)$ are given functions. Then the solution of (3.12) *under the given conditions, is given by*

$$\begin{aligned} U(x,t)=&\int_{-\infty}^{\infty} G_1(x-\xi,t)U_0(\xi)d\xi \\ &+b\int_0^t (t-\tau)^{\mu-1}\Big[\int_{-\infty}^{\infty} G_4(x-\xi,t-\tau)V(\xi,\tau)d\xi\Big]d\tau, \end{aligned} \tag{3.4}$$

where the Green function $G_1(x,t)$ is given by (2.5) *and the function $G_4(x,t)$ is given by*

$$\begin{aligned} G_4(x,t)=&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\mu,\mu}(-a|k|^{\alpha}t^{\mu})dk \\ =&\frac{1}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{a^{1/\alpha}t^{\mu/\alpha}}\left| \begin{array}{c} (1,1/\alpha),(\mu,\mu/\alpha),\left(1,\frac{1}{2}\right) \\ (1,1/\alpha),(1,1),\left(1,\frac{1}{2}\right) \end{array}\right.\right], \end{aligned} \tag{3.5}$$

where $a=i\hbar/2m$.

(b) If we put $b=0$, the above corollary give rise to the solution of one dimensional space-time fractional Schrödinger equation as obtained by Saxena et al. [27].

(c) Again, if we set $a=c^2$ and $\beta=0$, Theorem 2.1 yields to the solution of non-homogeneous generalized fractional diffusion-wave equation as follows:

Corollary 3.2. *Consider one dimensional non-homogeneous generalized fractional diffusion-wave equation, defined by*

$$ {}_0D_t^{\mu,\nu}U(x,t)=c^2\Delta^{\alpha/2}U(x,t)+bV(x,t), \quad (x\in\mathbb{R}, \quad t>0, \quad 0<\alpha\le 2), \tag{3.6}$$

with initial conditions

$$ I_{0+}^{(1-\nu)(1-\mu)}U(x,0+)=U_0(x), \quad (-\infty<x<\infty, \quad 0<\mu<1, \quad 0\le\nu\le 1), \tag{3.7}$$

and

$$ U(x,t)\to 0 \ \text{ as } \ |x|\to\infty, \tag{3.8}$$

where b is arbitrary, and $U_0(x)$ and $V(x,t)$ are given functions. Then the solution of (3.6) *under the given conditions, is given by*

$$ \begin{aligned} U(x,t)=&\int_{-\infty}^{\infty} G_1(x-\xi,t)U_0(\xi)d\xi \\ &+b\int_0^t (t-\tau)^{\mu-1}\Big[\int_{-\infty}^{\infty} G_4(x-\xi,t-\tau)V(\xi,\tau)d\xi\Big]d\tau, \end{aligned} \tag{3.9}$$

where the Green function $G_1(x,t)$ and $G_4(x,t)$ are respectively given by (2.5) *and* (3.5) *with $a=c^2$.*

(d) If we set $\nu=0$, then the generalized fractional derivative (1.11) reduces to the classical Riemann-Liouville fractional derivative operator (1.6) and the Theorems 2.1 and 2.2 give rise to the following results:

Corollary 3.3. *Consider the following one dimensional non-homogeneous generalized fractional partial differential equation, defined by*

$$ {}_0D_t^{\mu}U(x,t)=a\Delta^{\alpha/2}U(x,t)+b\Delta^{\beta/2}V(x,t), \quad (x\in\mathbb{R}, \ t>0, \ 0<\alpha\le 2, \ 0<\beta\le 2), \tag{3.10}$$

with initial conditions

$$ {}_0D_t^{(\mu-1)}U(x,0)=U_0(x); {}_0D_t^{(\mu-2)}U(x,0)=0, \quad (-\infty<x<\infty, \quad 0<\mu<2), \tag{3.11}$$

and

$$ U(x,t)\to 0 \ \text{ as } \ |x|\to\infty, \tag{3.12}$$

where the fractional derivative ${}_0D_t^{\mu}$ *with respect to t of Riemann-Liouville type is given by* (1.6) *and* $\Delta^{\alpha/2}$ *is the fractional generalization of the Laplace operator, defined by* (1.13). *a and b are arbitrary constants,* $U_0(x)$ *and* $V(x,t)$ *are given functions. Then the solution of* (3.10) *under the given conditions, is given by*

$$\begin{aligned} U(x,t)=&\int_{-\infty}^{\infty} G_5(x-\xi,t)U_0(\xi)d\xi \\ &-b\int_0^t (t-\tau)^{\mu-1}\Big[\int_{-\infty}^{\infty} G_2(x-\xi,t-\tau)V(\xi,\tau)d\xi\Big]d\tau, \end{aligned} \tag{3.13}$$

where the Green function $G_5(x,t)$ *is given by*

$$G_5(x,t)=\frac{t^{\mu-1}}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{a^{1/\alpha}t^{\mu/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(\mu,\mu/\alpha),\left(1,\frac{1}{2}\right)\\(1,1/\alpha),(1,1),\left(1,\frac{1}{2}\right)\end{matrix}\right], \tag{3.14}$$

and the function $G_2(x,t)$ *is given by* (2.6).

Corollary 3.4. *Consider the following one dimensional fractional differential equation*

$${}_0D_t^{\mu}U(x,t)=a\Delta^{\alpha/2}U(x,t)+b\Delta^{\beta/2}U(x,t),\quad (x\in\mathbb{R},\ t>0,\ 0<\alpha\le 2,\ 0<\beta\le 2), \tag{3.15}$$

with initial conditions

$${}_0D_t^{(\mu-1)}U(x,0)=U_0(x);{}_0D_t^{(\mu-2)}U(x,0)=0,\quad (-\infty<x<\infty,\quad 0<\mu<2), \tag{3.16}$$

and

$$U(x,t)\to 0 \ \text{as}\ |x|\to\infty, \tag{3.17}$$

where a and b are arbitrary constants, and $U_0(x)$ *is a given function, then for solution of* (3.15) *under the given conditions, there holds the formula*

$$U(x,t)=\int_{-\infty}^{\infty} G_6(x-\xi,t)U_0(\xi)d\xi, \tag{3.18}$$

where the Green function $G_6(x,t)$ *is given by*

$$G_6(x,t)=\frac{t^{\mu-1}}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}E_{\mu,\mu}[-(a|k|^{\alpha}+b|k|^{\beta})t^{\mu}]dk. \tag{3.19}$$

(e) If we set $\nu=1$, the Hilfer fractional derivative reduces to the Caputo fractional derivative operator (1.9) and the Theorems 2.1 and 2.2 yields the following results obtained by the author (in a joint paper) in a slightly different form [21]:

Corollary 3.5. *Consider the following one dimensional non-homogeneous generalized fractional partial differential equation, defined by*

$$ {}_0^C D_t^{\mu} U(x,t) = a\Delta^{\alpha/2} U(x,t) + b\Delta^{\beta/2} V(x,t), \quad (x\in\mathbb{R},\ t>0,\ 0<\alpha\le 2,\ 0<\beta\le 2,\ 0<\mu<1), \tag{3.20}$$

with initial conditions

$$U(x,0+) = U_0(x), \quad (-\infty<x<\infty), \tag{3.21}$$

and

$$U(x,t)\to 0 \ \text{ as } \ |x|\to\infty, \tag{3.22}$$

where a and b are arbitrary constants, $U_0(x)$ and $V(x,t)$ are given functions. Then the solution of (3.20) *under the given conditions, is given by*

$$\begin{aligned} U(x,t) = &\int_{-\infty}^{\infty} G_7(x-\xi,t) U_0(\xi) d\xi \\ &-b\int_0^t (t-\tau)^{\mu-1} \Big[\int_{-\infty}^{\infty} G_2(x-\xi,t-\tau) V(\xi,\tau) d\xi \Big] d\tau, \end{aligned} \tag{3.23}$$

where the Green function $G_7(x,t)$ is given by

$$G_7(x,t) = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{a^{1/\alpha} t^{\mu/\alpha}} \middle| \begin{array}{c} (1,1/\alpha),(1,\mu/\alpha),\left(1,\frac{1}{2}\right) \\ (1,1/\alpha),(1,1),\left(1,\frac{1}{2}\right) \end{array} \right], \tag{3.24}$$

and the function $G_2(x,t)$ is given by (2.6).

Corollary 3.6. *Consider the following one dimensional fractional differential equation*

$$ {}_0^C D_t^{\mu} U(x,t) = a\Delta^{\alpha/2} U(x,t) + b\Delta^{\beta/2} U(x,t), \ (x\in\mathbb{R},\ t>0,\ 0<\alpha\le 2,\ 0<\beta\le 2,\ 0<\mu<1), \tag{3.25}$$

with initial conditions

$$U(x,0+) = U_0(x), \quad (-\infty<x<\infty), \tag{3.26}$$

and

$$U(x,t)\to 0 \ \text{ as } \ |x|\to\infty, \tag{3.27}$$

where a and b are arbitrary constants, and $U_0(x)$ is a given function, then for solution of (3.25) *under the given conditions, there holds the formula*

$$U(x,t) = \int_{-\infty}^{\infty} G_8(x-\xi,t) U_0(\xi) d\xi, \tag{3.28}$$

where the Green function $G_8(x,t)$ is given by

$$G_8(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} E_{\mu,1}[-(a|k|^{\alpha} + b|k|^{\beta}) t^{\mu}] dk. \tag{3.29}$$

(a) Again, if we set $a=i\hbar/(2m)$ and $\beta=0$, Corollaries 3.5 and 3.6 yields to the solutions of non-homogeneous space-time fractional Schrödinger equations involving Riemann-Liouville and Caputo type fractional derivatives respectively. Which, on further setting $b=0$, reduced to the known results due to Saxena et al. (see [27], Corollary 2.1 and 2.2).

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