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Non-Semisimple Lie Algebras of Block Matrices and Applications to Bi-Integrable Couplings

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Abstract. We propose a class of non-semisimple matrix loop algebras consisting of 3×3 block matrices, and form zero curvature equations from the presented loop algebras to generate bi-integrable couplings. Applications are made for the AKNS soliton hierarchy and Hamiltonian structures of the resulting integrable couplings are constructed by using the associated variational identities.

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Key words: Bi-integrable couplings, non-semisimple matrix loop algebras, AKNS hierarchy, Hamiltonian structure, symmetry.

1 Introduction

For a given integrable system, integrable couplings are non-trivial larger systems which are still integrable and include the original integrable system as a sub-system. The concept of integrable couplings was systematically introduced in 1996 (see [16] for details), and since then it has been an attractive research topic of many publications (see, e.g., [7,8, 10,19,26–29,31,32]). A few methods of constructing integrable couplings have been developed, such as the perturbation method [8,15,16], enlarging spectral problems [10,11], and constructing new matrix loop Lie algebras [5,30]. Recently, a new class of non-semisimple matrix loop algebras was proposed in [21] for investigating nonlinear bi-integrable couplings.

In this paper, we will introduce 10 new classes of Lie algebras of 3×3 block matrices which can generate bi-integrable couplings.

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First, let us recall the problem of integrable couplings: for a given integrable system of evolution equations:

$$u_t = K(u), \tag{1.1}$$

where u is in some manifold M and K is a suitable C^∞ vector field on M, we look for an enlarged non-trivial integrable system which includes the original system as a subsystem. It is known that a change of the arrangement of equations in a system does not lose integrability of the system, and therefore we study how to construct an enlarged non-trivial system of evolution equations of the triangular form. Such a bi-integrable coupling of the system (1.1) is defined as follows [21]:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u,u_1), \\ u_{2,t} = S_2(u,u_1,u_2), \end{cases} \tag{1.2}$$

where u_1 and u_2 are new dependent variables, and S_1 and S_2 are vector fields depending on the indicated variables. We call this integrable system a nonlinear coupling if at least one of $S_1(u,u_1)$ and $S_2(u,u_1,u_2)$ is nonlinear with respect to the sub-vectors u_1,u_2 of dependent variables.

In this paper, we will introduce new non-semisimple Lie algebras of 3×3 block matrices in Section 2, and then in Section 3, we will describe a general scheme to construct bi-integrable couplings associated with the newly presented Lie algebras. Section 4 is devoted to applications to the AKNS hierarchy and mathematical structures that the resulting bi-integrable couplings possess, such as infinitely many symmetries, infinitely many conserved functionals, and bi-Hamiltonian structures.

2 Loop algebras of 3×3 block matrices

We seek for non-semisimple matrix Lie algebras, under which we can generate bi-integrable couplings of an integrable system (1.1) by using the zero curvature equation. First, we look for matrix algebras consisting of 3×3 block matrices of the form

$$M(A_1,A_2,A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & \sum_{i=1}^{3} \alpha_{1,i} A_i & \sum_{i=1}^{3} \alpha_{2,i} A_i \\ 0 & 0 & \sum_{i=1}^{3} \alpha_{3,i} A_i \end{bmatrix},$$

where $\alpha_{i,j}$, $1 \le i,j \le 3$ are constants to be determined. The reason why we choose these triangular type block matrices is that Lax pair [6] matrices U and V of triangular types will help generate bi-integrable couplings. Thus in the next step, we want to classify classes of such matrices which form matrix Lie algebras under matrix commutator

$$[U,V] := UV - VU. \tag{2.1}$$

As a result, we require that the Lie bracket

$$[M(A_1,A_2,A_3),M(B_1,B_2,B_3)]$$

of block matrices $M(A_1,A_2,A_3)$ and $M(B_1,B_2,B_3)$ must be of the form $M(C_1,C_2,C_3)$ for certain square submatrices C_1,C_2,C_3 of the same order as A_i and B_i, $1\le i\le 3$. It thus follows that such square submatrices C_1, C_2 and C_3 read

$$\begin{cases} C_1=[A_1,B_1], \\ C_2=[A_1,B_2]+\alpha_{1,1}[A_2,B_1], \\ C_3=[A_1,B_3]+\alpha_{2,1}[A_2,B_1]+\alpha_{2,2}[A_2,B_2]+\alpha_{2,3}[A_2,B_3] \\ \qquad +\alpha_{3,1}[A_3,B_1]+\alpha_{3,2}[A_3,B_2]+\alpha_{3,3}[A_3,B_3]. \end{cases} \tag{2.2}$$

A direct Maple computation shows that there are many classes of non-semisimple Lie algebras of such matrices. Here is a list of them:

$$\text{Class}_{\mathbf{1}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1+\alpha A_2+\beta A_3 & 0 \\ 0 & 0 & A_1+\alpha A_2+\beta A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{2}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1+\dfrac{\beta}{\alpha}A_2 & \alpha A_1+\beta A_2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{Class}_{\mathbf{3}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1+\alpha A_2 & \beta A_1+\alpha A_3 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{Class}_{\mathbf{4}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1+\alpha A_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{Class}_{\mathbf{5}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1+\alpha A_2 & \beta A_2+\gamma A_3 \\ 0 & 0 & A_1+\gamma A_2-\dfrac{\gamma(\alpha-\gamma)}{\beta}A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{6}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1+\alpha A_2 & 0 \\ 0 & 0 & A_1+\beta A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{7}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1+\alpha A_2 & \alpha A_3 \\ 0 & 0 & A_1 \end{bmatrix},$$

$$\text{Class}_{\mathbf{8}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1+\alpha A_2 & \alpha A_3 \\ 0 & 0 & A_1+\alpha A_2+\beta A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{9}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & \alpha A_1+\alpha^2\beta A_2+\alpha\beta A_3 \\ & 0 & A_1+\alpha\beta A_2+\beta A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{10}}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where α, β, γ are arbitrarily fixed constants.

We shall focus on one class of the presented non-semisimple Loop matrix Lie algebras and construct bi-integrable couplings by using the enlarged zero curvature equation. Moreover, the resulting bi-integrable couplings have infinitely many symmetries and conserved functionals, which further indicates that they often possess bi-Hamiltonian structures.

In what follows, we consider a class of triangular block matrices

$$M(A_1,A_2,A_3)=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & \alpha A_2+\alpha\beta A_3 \\ 0 & 0 & A_1+\alpha\beta A_2+\alpha\beta^2 A_3 \end{bmatrix}, \tag{2.3}$$

where A_1, A_2, A_3 are square matrices of the same order and α,β are arbitrarily fixed constants. This class of triangular block matrices is a special case of Class_5, if we set α in Class_5 to be zero. Obviously, under the matrix Lie bracket $[\cdot,\cdot]$ as defined in (2.1), all block matrices M_1, M_2 as defined in (2.3) form a matrix Lie algebra, since for any square matrices A_1, A_2, A_3 and B_1, B_2, B_3 of the same order, we have

$$[M(A_1,A_2,A_3),M(B_1,B_2,B_3)]=M(C_1,C_2,C_3), \tag{2.4}$$

with

$$\begin{cases} C_1=[A_1,B_1], \\ C_2=[A_1,B_2]+[A_2,B_1], \\ C_3=[A_1,B_3]+\alpha[A_2,B_2]+\alpha\beta[A_2,B_3]+[A_3,B_1]+\alpha\beta[A_3,B_2]+\alpha\beta^2[A_3,B_3]. \end{cases}$$

Up to this point, we have not specified what the square matrices A_1, A_2, A_3 will be taken. In the next step, we will concentrate on this matrix Lie algebra and take its decomposition as a semi-direct sum of two subalgebras.

We define two matrix loop Lie algebras

$$g_1=\{M(A_1,0,0)|\text{entries of } A_1-\text{Laurent series in } \lambda\}, \tag{2.5}$$

and

$$g_2=\{M(0,A_2,A_3)|\text{entries of } A_2,A_3-\text{Laurent series in } \lambda\}. \tag{2.6}$$

Next, we take a semi-direct sum

$$\bar{g}=g_1 \Subset g_2 \tag{2.7}$$

of these two Lie algebras g_1 and g_2 as introduced in (2.5) and (2.6) to get

$$\bar{g}=\{M(A_1,A_2,A_3)|\text{entries of } A_1,A_2,A_3-\text{Laurent series in } \lambda\}. \tag{2.8}$$

It follows that $\bar{g}$ is an infinite-dimensional Lie algebra. The notion of semi-direct sums $\bar{g}=g_1 \Subset g_2$ means that the two subalgebras g_1 and g_2 satisfy

$$[g_1,g_2]\subseteq g_2,$$

where $[g_1,g_2]=\{[M_1,M_2]|M_1\in g_1,\ M_2\in g_2\}$. Obviously, g_2 is an ideal Lie sub-algebra of $\bar{g}$. We also have the closure property between g_1 and g_2 under the matrix multiplication

$$g_1g_2,g_2g_1\subseteq g_2, \tag{2.9}$$

where $g_1g_2=\{AB|A\in g_1,\ B\in g_2\}$, $g_2g_1=\{AB|A\in g_2,\ B\in g_1\}$, to guarantee that a zero curvature equation over semi-direct sums of Lie algebras can generate discrete coupling systems [11,19–21].

Now we have constructed the non-semisimple Lie algebra, associated with which we will formulate a scheme for constructing bi-integrable couplings.

3 A general scheme for constructing bi-integrable couplings

In order to take advantage of zero curvature equations associated with the semi-direct sum of Lie algebras, we assume that the original integrable system

$$u_t=K(u)$$

is determined by a zero curvature equation

$$U_t-V_x+[U,V]=0, \tag{3.1}$$

where the Lax pair $U=U(u,\lambda)$ and $V=V(u,\lambda)$, with λ being the spectral parameter, are square matrices belonging to some semisimple matrix Lie algebra [2].

Our goal is to construct bi-integrable couplings

$$\begin{cases} u_t=K(u), \\ u_{1,t}=S_1(u,u_1), \\ u_{2,t}=S_2(u,u_1,u_2), \end{cases}$$

of the system (1.1), and therefore we enlarge the original spectral matrix U and define the corresponding enlarged spectral matrix $\bar{U}$ as follows:

$$\bar{U}=\bar{U}(\bar{u},\lambda)=M(U(u,\lambda),U_1(u_1,\lambda),U_2(u_2,\lambda))\in\bar{g}=g_1\Subset g_2, \tag{3.2}$$

where $\bar{u}=(u^T,u_1^T,u_2^T)^T$. We also assume that its enlarged Lax matrix $\bar{V}$ is in the form of

$$\bar{V}=\bar{V}(\bar{u},\lambda)=M(V(u,\lambda),V_1(u,u_1,\lambda),V_2(u,u_1,u_2,\lambda))\in\bar{g}=g_1 \Subset g_2. \tag{3.3}$$

Apparently, the Lie bracket $[\bar{U},\bar{V}]$ of $\bar{U}$ and $\bar{V}$ is in $\bar{g}$.

Consequently, the corresponding enlarged zero curvature equation

$$\bar{U}_t-\bar{V}_x+[\bar{U},\bar{V}]=0 \tag{3.4}$$

is equivalent to the following triangle system

$$\begin{cases} U_t-V_x+[U,V]=0,\\ U_{1,t}-V_{1,x}+[U,V_1]+[U_1,V]=0,\\ U_{2,t}-V_{2,x}+[U,V_2]+\alpha[U_1,V_1]+\alpha\beta[U_1,V_2]+[U_2,V]+\alpha\beta[U_2,V_1]+\alpha\beta^2[U_2,V_2]=0. \end{cases} \tag{3.5}$$

The first equation above precisely gives the system (1.1), and the second and third equations give the sub-systems $u_{1,t}=S_1(u,u_1)$ and $u_{2,t}=S_2(u,u_1,u_2)$, respectively. Thus, the triangle system gives a bi-integrable coupling system (1.2). This shows a basic idea of constructing bi-integrable couplings by using the semi-direct sum of Lie algebras in (2.5) and (2.6).

We assume that we knew U, U_1 and U_2, and then we are going to seek for a polynomial solution $\bar{V}$ of (3.4) of degree m (hence we denote this $\bar{V}$ by $\bar{V}^{[m]}$ and its corresponding time variable by t_m).

The constructing scheme is stated as follows.

The first step of formulation of the hierarchy is to construct a generating function $\bar{W}$ by solving the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x=[\bar{U},\bar{W}],\quad \bar{W}=\bar{W}(\bar{u},\lambda), \tag{3.6}$$

with the following form

$$\bar{W}=M(W(u,\lambda),W_1(u,u_1,\lambda),W_2(u,u_1,u_2,\lambda))\in\bar{g}=g_1 \Subset g_2. \tag{3.7}$$

Plugging (3.7) into (3.6), we get the triangle system

$$\begin{cases} W_x=[U,W],\\ W_{1,x}=[U,W_1]+[U_1,W],\\ W_{2,x}=[U,W_2]+\alpha[U_1,W_1]+\alpha\beta[U_1,W_2]+[U_2,W]+\alpha\beta[U_2,W_1]+\alpha\beta^2[U_2,W_2]. \end{cases} \tag{3.8}$$

We assume that W, W_1, W_2 are in the form of

$$W=\sum_{i\geq 0}W_{0,i}\lambda^{-i},\quad W_1=\sum_{i\geq 0}W_{1,i}\lambda^{-i},\quad W_2=\sum_{i\geq 0}W_{2,i}\lambda^{-i}. \tag{3.9}$$

Then we define $\bar{V}^{[m]}$ by

$$\bar{V}^{[m]}=M(V^{[m]},V_1^{[m]},V_2^{[m]})\in\bar{g}=g_1 \Subset g_2, \tag{3.10}$$

and

$$V^{[m]}=(\lambda^{m+1}W)_{+}+\Delta_m,\quad V_i^{[m]}=(\lambda^{m+1}W_i)_{+}+\Delta_{m,i},\quad i=1,2,\quad m\geq 0, \tag{3.11}$$

where $(\lambda^{m+1}P)_{+}$ denotes the polynomial part of $\lambda^{m+1}P$ in λ, and choose $\Delta_{m,i}$ to make sure that (3.4) with $\bar{V}^{[m]}$, $m\geq 0$, i.e.,

$$\bar{U}_{t_m}-\bar{V}_x^{[m]}+[\bar{U},\bar{V}^{[m]}]=0,\quad m\geq 0, \tag{3.12}$$

generate a soliton hierarchy of bi-integrable coupling systems

$$\bar{u}_{t_m}=\bar{K}_m(\bar{u}),\quad m\geq 0, \tag{3.13}$$

where

$$\bar{u}=\left[\begin{array}{c} u \\ u_1 \\ u_2 \end{array}\right],\quad \bar{K}_m(\bar{u})=\left[\begin{array}{c} K_m(u) \\ S_{1,m}(u,u_1) \\ S_{2,m}(u,u_1,u_2) \end{array}\right],\qquad m\geq 0. \tag{3.14}$$

We shall apply this scheme to the AKNS soliton hierarchy to construct its bi-integrable couplings in the next section.

4 Applications to the AKNS hierarchy

4.1 The AKNS hierarchy

We consider the AKNS soliton hierarchy [1,14]. Its spectral problem is given by

$$\phi_x=U\phi,\quad U=U(u,\lambda)=\left[\begin{array}{cc} -\lambda & p \\ q & \lambda \end{array}\right],\quad u=\left[\begin{array}{c} p \\ q \end{array}\right],\quad \phi=\left[\begin{array}{c} \phi_1 \\ \phi_2 \end{array}\right]. \tag{4.1}$$

If we consider the stationary zero curvature equation

$$W_x=[U,W], \tag{4.2}$$

and assume that a solution W solution to (4.2) is in the form of

$$W=\left[\begin{array}{cc} a & b \\ c & -a \end{array}\right]=\sum_{i\geq 0}W_{0,i}\lambda^{-i}=\sum_{i\geq 0}\left[\begin{array}{cc} a_i & b_i \\ c_i & -a_i \end{array}\right]\lambda^{-i}. \tag{4.3}$$

By plugging (4.3) in (4.2), we obtain

$$\left\{\begin{array}{l} a_x=pc-qb, \\ b_x=-2\lambda b-2pa, \\ c_x=2qa+2\lambda c. \end{array}\right.$$

Comparing the coefficient of each λ^{-i}, $i\geq 0$, we get

$$\left\{\begin{array}{l} a_{i,x}=pc_i-qb_i, \\ b_{i,x}=-2b_{i+1}-2pa_i, \quad \text{for} \quad i\geq 0, \\ c_{i,x}=2qa_i+2c_{i+1}, \end{array}\right. \tag{4.4}$$

i.e.,

$$\left\{\begin{array}{l} a_{i+1,x}=pc_{i+1}-qb_{i+1}, \\ b_{i+1}=-\dfrac{1}{2}b_{i,x}-pa_i, \quad \text{for} \quad i\geq 0. \\ c_{i+1}=\dfrac{1}{2}c_{i,x}-qa_i, \end{array}\right. \tag{4.5}$$

By the condition on the coefficient of λ, we assume

$$a_0=-1, \quad b_0=c_0=0, \tag{4.6}$$

and then the first three sequences can be obtained as follows:

$$\left\{\begin{array}{l} b_1=p, \quad c_1=q, \quad a_1=0, \\ b_2=-\dfrac{1}{2}p_x, \quad c_2=\dfrac{1}{2}q_x, \quad a_2=\dfrac{1}{2}pq, \\ b_3=\dfrac{1}{4}p_{xx}-\dfrac{1}{2}p^2q, \quad c_3=\dfrac{1}{4}q_{xx}-\dfrac{1}{2}pq^2, \quad a_3=\dfrac{1}{4}(pq_x-p_xq). \end{array}\right.$$

We form the zero curvature equations

$$U_{t_m}-V_x^{[m]}+[U,V^{[m]}]=0, \quad V^{[m]}=(\lambda^m W)_+, \quad m\geq 0, \tag{4.7}$$

to generate the AKNS hierarchy of soliton equations:

$$u_{t_m}=K_m=\left[\begin{array}{c} -2b_{m+1} \\ 2c_{m+1} \end{array}\right]=\Phi^m\left[\begin{array}{c} -2p \\ 2q \end{array}\right]=J\frac{\delta\mathcal{H}_m}{\delta u}, \quad m\geq 0, \tag{4.8}$$

with the Hamiltonian operator J, the hereditary recursion operator Φ and the Hamiltonian functionals being defined by

$$J=\left[\begin{array}{cc} 0 & -2 \\ 2 & 0 \end{array}\right], \quad \Phi=\left[\begin{array}{cc} -\dfrac{1}{2}\partial+p\partial^{-1}q & p\partial^{-1}p \\ -q\partial^{-1}q & \dfrac{1}{2}\partial-q\partial^{-1}p \end{array}\right], \quad \partial=\frac{\partial}{\partial x}, \tag{4.9a}$$

$$\mathcal{H}_m=\int\frac{2a_{m+2}}{m+1}dx, \quad m\geq 0. \tag{4.9b}$$

We will enlarge the zero curvature equations to construct bi-integrable couplings in the following subsection.

4.2 Bi-integrable couplings

For the AKNS hierarchy spectral problem (4.1), using the matrix Lie algebra (2.8) we have chosen in the last section, we define the corresponding enlarged spectral matrix by

$$\bar{U}=\bar{U}(\bar{u},\lambda)=M(U,U_1,U_2)\in\bar{g}=g_1 \Subset g_2, \tag{4.10a}$$

$$U_1=U_1(u_1)=\begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix}, \quad U_2=U_2(u_2)=\begin{bmatrix} 0 & v \\ w & 0 \end{bmatrix}, \tag{4.10b}$$

where $\bar{u}=(u^T,u_1^T,u_2^T)^T$, $u_1=(r,s)^T$, $u_2=(v,w)^T$, and r, s, v, w are new dependent variables.

To solve the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x=[\bar{U},\bar{W}], \tag{4.11}$$

we set a solution of the following form

$$\bar{W}=M(W,W_1,W_2)\in\bar{g}=g_1 \Subset g_2, \tag{4.12}$$

and assume that W as defined in (4.3),

$$W_1,W_2\in\widetilde{\mathrm{sl}}(2,\mathbb{R})=\{A\in\mathrm{sl}(2,\mathbb{R})\,|\text{ entries of } A-\text{Laurent series in }\lambda\}$$

are in the form of

$$\begin{cases} W_1=W_1(u,u_1,\lambda)=\begin{bmatrix} e & f \\ g & -e \end{bmatrix}=\sum_{i\geq 0}\begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix}\lambda^{-i}, \\ W_2=W_2(u,u_1,u_2,\lambda)=\begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix}=\sum_{i\geq 0}\begin{bmatrix} e'_i & f'_i \\ g'_i & -e'_i \end{bmatrix}\lambda^{-i}. \end{cases}$$

It now follows from the enlarged stationary zero curvature equation (4.11) that

$$\begin{cases} W_x=[U,W], \\ W_{1,x}=[U,W_1]+[U_1,W], \\ W_{2,x}=[U,W_2]+[U_2,W]+\alpha[U_1,W_1]+\alpha\beta[U_1,W_2]+\alpha\beta[U_2,W_1]+\alpha\beta^2[U_2,W_2]. \end{cases} \tag{4.13}$$

The above equation system equivalently leads to

$$\begin{cases} a_x=-2c\lambda+2qb, \\ b_x=2qa-2cr, \\ c_x=2a\lambda-2br, \end{cases} \qquad \begin{cases} e_x=pg+rc-qf-sb, \\ f_x=-2\lambda f-2pe-2ra, \\ g_x=2qe+2\lambda g+2sa, \end{cases}$$

$$\begin{cases} e'_x=-wb+vc-\alpha(s+\beta w)f+\alpha(r+\beta v)g-(q+\alpha\beta s+\alpha\beta^2 w)f'+(p+\alpha\beta r+\alpha\beta^2 v)g', \\ f'_x=-2av-2\alpha(r+\beta v)e-2(p+\alpha\beta r+\alpha\beta^2 v)e'-2\lambda f', \\ g'_x=2aw+2\alpha(s+\beta w)e+2(q+\alpha\beta s+\alpha\beta^2 w)e'+2\lambda g'. \end{cases}$$

By assuming

$$e=\sum_{i\geq 0} e_i\lambda^{-i},\quad f=\sum_{i\geq 0} f_i\lambda^{-i},\quad g=\sum_{i\geq 0} g_i\lambda^{-i}, \tag{4.14}$$

and

$$e'=\sum_{i\geq 0} e'_i\lambda^{-i},\quad f'=\sum_{i\geq 0} f'_i\lambda^{-i},\quad g'=\sum_{i\geq 0} g'_i\lambda^{-i}, \tag{4.15}$$

and comparing the coefficient of each λ^{-i}, $i\geq 0$, we obtain

$$\begin{cases} f_{i+1}=-\dfrac{1}{2}f_{ix}-pe_i-ra_i, \\ g_{i+1}=\dfrac{1}{2}g_{ix}-qe_i-sa_i, \\ e_{i+1,x}=pg_{i+1}+rc_{i+1}-qf_{i+1}-sb_{i+1}, \end{cases} \quad \text{for } i\geq 0, \tag{4.16a}$$

$$\begin{cases} f'_{i+1}=-\dfrac{1}{2}f'_{ix}-(p+\alpha\beta r+\alpha\beta^2 v)e'_i-\alpha(r+\beta)ve_i-va_i, \\ g'_{i+1}=\dfrac{1}{2}g'_{ix}-(q+\alpha\beta s+\alpha\beta^2 w)e'_i-\alpha(s+\beta)we_i-wa_i, \\ e'_{i+1,x}=-wb_{i+1}+vc_{i+1}-\alpha(s+\beta w)f_{i+1}+\alpha(r+\beta v)g_{i+1} \\ \qquad\quad -(q+\alpha\beta s+\alpha\beta^2 w)f'_{i+1}+(p+\alpha\beta r+\alpha\beta^2 v)g'_{i+1}, \end{cases} \quad \text{for } i\geq 0. \tag{4.16b}$$

Then the recursion relations (4.16a) and (4.16b) generate the sequences of $\{e_i\}_{i\geq 1}$, $\{f_i\}_{i\geq 1}$, $\{g_i\}_{i\geq 1}$ and $\{e'_i\}_{i\geq 1}$, $\{f'_i\}_{i\geq 1}$, $\{g'_i\}_{i\geq 1}$.

Upon introducing

$$e_0=e'_0=-1,\quad f_0=g_0=f'_0=g'_0=0, \tag{4.17}$$

to satisfy the conditions on the coefficients of λ in (4.2), we can compute the first few sets as follows:

$$\begin{cases} e_1=0, \\ f_1=p+r, \\ g_1=q+s, \end{cases} \tag{4.18a}$$

$$\begin{cases} e_2=\dfrac{1}{2}pq+\dfrac{1}{2}sp+\dfrac{1}{2}rq, \\ f_2=-\dfrac{1}{2}p_x-\dfrac{1}{2}r_x, \\ g_2=\dfrac{1}{2}q_x+\dfrac{1}{2}s_x, \end{cases} \tag{4.18b}$$

$$\begin{cases} e_3=\dfrac{1}{4}pq_x+\dfrac{1}{4}s_xp-\dfrac{1}{4}qp_x-\dfrac{1}{4}sp_x-\dfrac{1}{4}r_xq+\dfrac{1}{4}rq_x, \\ f_3=\dfrac{1}{4}p_{xx}+\dfrac{1}{4}r_{xx}-\dfrac{1}{2}p^2q-\dfrac{1}{2}sp^2-rpq, \\ g_3=\dfrac{1}{4}q_{xx}+\dfrac{1}{4}s_{xx}-\dfrac{1}{2}q^2p-\dfrac{1}{2}rq^2-spq, \end{cases} \tag{4.18c}$$

and

$$\begin{cases} e_1' = 0, \\ f_1' = p + \alpha(1+\beta)r + (1+\alpha\beta+\alpha\beta^2)v, \\ g_1' = q + \alpha(1+\beta)s + (1+\alpha\beta+\alpha\beta^2)w, \end{cases} \tag{4.19a}$$

$$\begin{cases} e_2' = \frac{1}{2}pq + \frac{1}{2}\alpha(1+\beta)ps + \frac{1}{2}(1+\alpha\beta+\alpha\beta^2)\big(pw+rq+vq+\alpha rs+\alpha\beta rw+\alpha\beta vs+\alpha\beta^2 vw\big), \\ f_2' = -\frac{1}{2}p_x - \frac{1}{2}\alpha(1+\beta)r_x - \frac{1}{2}(1+\alpha\beta+\alpha\beta^2)v_x, \\ g_2' = \frac{1}{2}q_x + \frac{1}{2}\alpha(1+\beta)s_x + \frac{1}{2}(1+\alpha\beta+\alpha\beta^2)w_x, \end{cases} \tag{4.19b}$$

$$\begin{cases} e_3' = \frac{1}{4}(pq_x - qp_x) + \frac{1}{4}\alpha(1+\beta)(ps_x - p_x s + rq_x - r_x q) + \frac{1}{4}(1+\alpha\beta+\alpha\beta^2)\big[(pw_x - p_x w) \\ \quad +\alpha(rs_x - r_x s + rw_x - r_x w - v_x q + vq_x + vs_x - v_x s + vw_x - v_x w)\big], \\ f_3' = \frac{1}{4}p_{xx} - \frac{1}{2}p^2 q + \alpha(1+\beta)\Big(\frac{1}{4}r_{xx} - \frac{1}{2}p^2 s - prq\Big) + (1+\alpha\beta+\alpha\beta^2)\Big[\frac{1}{4}v_{xx} - pvq \\ \quad -\frac{1}{2}p^2 w - \alpha\Big(prs + \frac{1}{2}r^2 q\Big) - \alpha\beta(prw + pvs + rvq) - \alpha\beta^2\Big(pvw + \frac{1}{2}v^2 q\Big) - \frac{1}{2}\alpha^2\beta r^2 s \\ \quad -\alpha^2\beta^2\Big(rvs + \frac{1}{2}r^2 w\Big) - \alpha^2\beta^3\Big(rvw + \frac{1}{2}v^2 s\Big) - \frac{1}{2}\alpha^2\beta^4 v^2 w\Big], \\ g_3' = \frac{1}{4}q_{xx} - \frac{1}{2}q^2 p + \alpha(1+\beta)\Big(\frac{1}{4}s_{xx} - \frac{1}{2}rq^2 - qps\Big) + (1+\alpha\beta+\alpha\beta^2)\Big[\frac{1}{4}w_{xx} - pqw - \frac{1}{2}vq^2 \\ \quad -\alpha\Big(qrs + \frac{1}{2}s^2 p\Big) - \alpha\beta(qrw + qvs + spw) - \alpha\beta^2\Big(\frac{1}{2}w^2 p + qvw\Big) - \frac{1}{2}\alpha^2\beta s^2 r \\ \quad -\alpha^2\beta^2\Big(\frac{1}{2}s^2 v + srw\Big) - \alpha^2\beta^3\Big(svw + \frac{1}{2}w^2 r\Big) - \frac{1}{2}\alpha^2\beta^4 w^2 v\Big]. \end{cases} \tag{4.19c}$$

Let us now define

$$\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}) \in \bar{g} = g_1 \Subset g_2, \tag{4.20}$$

and

$$\begin{cases} V_1^{[m]} = (\lambda^m V_1)_+ + \Delta_{m,1}, \\ V_2^{[m]} = (\lambda^m V_2)_+ + \Delta_{m,2}, \end{cases} \quad m \geq 0, \tag{4.21}$$

where $V^{[m]}$ is defined as in (4.7), and $\Delta_{m,i}$ are chosen as the zero matrix. Then, the m-th enlarged zero curvature equation

$$\bar{U}_{t_m} = \bar{V}_x^{[m]} - [\bar{U}, \bar{V}^{[m]}] \tag{4.22}$$

gives rise to

$$\begin{cases} U_{t_m} = V_x^{[m]} - [U, V^{[m]}], \\ U_{1,t_m} = V_{1,x}^{[m]} - [U, V_1^{[m]}] - [U_1, V^{[m]}], \\ U_{2,t_m} = V_{2,x}^{[m]} - [U, V_2^{[m]}] - [U_2, V^{[m]}] - \alpha[U_1, V_1^{[m]}] \\ \qquad -\alpha\beta[U_1, V_2^{[m]}] - \alpha\beta[U_2, V_1^{[m]}] - \alpha\beta^2[U_2, V_2^{[m]}]. \end{cases} \tag{4.23}$$

Thus, a hierarchy of coupling systems are generated for the AKNS hierarchy (4.7):

$$\bar{u}_{t_m}=\begin{bmatrix} p_{t_m}\\ q_{t_m}\\ r_{t_m}\\ s_{t_m}\\ v_{t_m}\\ w_{t_m}\end{bmatrix}=\bar{K}_m(\bar{u})=\begin{bmatrix} K_m(u)\\ S_{1,m}(u,u_1)\\ S_{2,m}(u,u_1,u_2)\end{bmatrix}=\begin{bmatrix} -2b_{m+1}\\ 2c_{m+1}\\ -2f_{m+1}\\ 2g_{m+1}\\ -2f'_{m+1}\\ 2g'_{m+1}\end{bmatrix},\quad m\geq 0. \tag{4.24}$$

This suggests that (4.24) provides a hierarchy of nonlinear bi-integrable couplings for the AKNS hierarchy of soliton equations. The first nonlinear bi-integrable coupling system reads

$$\begin{cases} p_{t_2}-\frac{1}{2}p_{xx}+p^2q,\\ q_{t_2}=\frac{1}{2}q_{xx}-pq^2,\\ r_{t_2}=-\frac{1}{2}p_{xx}-\frac{1}{4}r_{xx}+p^2q+sp^2+2rpq,\\ s_{t_2}=\frac{1}{2}q_{xx}+\frac{1}{2}s_{xx}-q^2p-rq^2-2spq,\\ v_{t_2}=-\frac{1}{2}p_{xx}+p^2q-2\alpha(1+\beta)\Big(\frac{1}{4}r_{xx}-\frac{1}{2}p^2s-prq\Big)\\ \qquad -2(1+\alpha\beta+\alpha\beta^2)\Big[\frac{1}{4}v_{xx}-pvq-\frac{1}{2}p^2w-\alpha\Big(prs+\frac{1}{2}r^2q\Big)\\ \qquad -\alpha\beta(prw+pvs+rvq)-\alpha\beta^2\Big(pvw+\frac{1}{2}v^2q\Big)-\frac{1}{2}\alpha^2\beta r^2s\\ \qquad -\alpha^2\beta^2\Big(rvs+\frac{1}{2}r^2w\Big)-\alpha^2\beta^3\Big(rvw+\frac{1}{2}v^2s\Big)-\frac{1}{2}\alpha^2\beta^4v^2w\Big],\\ w_{t_2}=\frac{1}{2}q_{xx}-q^2p+2\alpha(1+\beta)\Big(\frac{1}{4}s_{xx}-\frac{1}{2}rq^2-qps\Big)\\ \qquad +2(1+\alpha\beta+\alpha\beta^2)\Big[\frac{1}{4}w_{xx}-pqw-\frac{1}{2}vq^2-\alpha\Big(qrs+\frac{1}{2}s^2p\Big)\\ \qquad -\alpha\beta(qrw+qvs+spw)-\alpha\beta^2\Big(\frac{1}{2}w^2p+qvw\Big)-\frac{1}{2}\alpha^2\beta s^2r\\ \qquad -\alpha^2\beta^2\Big(\frac{1}{2}s^2v+srw\Big)-\alpha^2\beta^3\Big(svw+\frac{1}{2}w^2r\Big)-\frac{1}{2}\alpha^2\beta^4w^2v. \end{cases} \tag{4.25}$$

Refs. [8,9] formulated integrable couplings for given integrable systems by perturbations, in which the second component of the enlarged system was just the linearized system of the original system $u_t=K(u)$, while the bi-integrable couplings constructed above are nonlinear, because the third sub-systems are nonlinear.

4.3 Hamiltonian structures

It is known that when acting on non-semisimple Lie algebras, the Killing form is always degenerate, and, the trace identity (see [24, 25] for details) will not apply in this case. To solve this problem, the variational identity was introduced in [12, 13] under more general bilinear forms, which do not require the invariance property under an isomorphism of the Lie algebra. In this section, in order to generate Hamiltonian structures of the resulting bi-integrable couplings on the presented non-semisimple Lie algebra, we use the corresponding variational identity [13]:

$$\frac{\delta}{\delta\bar{u}}\int\langle\bar{W},\bar{U}_{\lambda}\rangle dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}(\lambda^{\gamma}\langle\bar{W},\bar{U}_{\bar{u}}\rangle), \tag{4.26}$$

where $\langle\cdot,\cdot\rangle$ is a required bilinear form, which is symmetric, non-degenerate, and invariant under the Lie bracket.

Let us now construct general bilinear forms with the symmetric, invariant, and non-degenerate properties $\langle\cdot,\cdot\rangle$ on $\bar{g}$. First, we transform the semi-direct sum $\bar{g}$ into a vector form via defining:

$$\sigma:\bar{g}\to\mathbb{R}^9,\ \ A\mapsto(a_1,\cdots,a_9)^T, \tag{4.27}$$

where

$$A=A(a_1,\cdots,a_9)=M(A_1,A_2,A_3),\quad A_i=\begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2}\end{bmatrix},\quad 1\le i\le 3. \tag{4.28}$$

This mapping σ induces a Lie algebraic structure on $\mathbb{R}^9$, which is isomorphic to the matrix loop algebra $\bar{g}$. Next we define the corresponding Lie bracket $[\cdot,\cdot]$ on $\mathbb{R}^9$ by

$$[a,b]^T=a^TR(b), \tag{4.29}$$

for any $a=(a_1,\cdots,a_9)^T$, $b=(b_1,\cdots,b_9)^T\in\mathbb{R}^9$, and

$$R(b)=M(R_1,R_2,R_3), \tag{4.30}$$

where R_1, R_2, and R_3 are the matrices defined by

$$R_i=\begin{bmatrix} 0 & 2b_{3i-1} & -2b_{3i} \\ b_{3i} & -2b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2b_{3i-2}\end{bmatrix},\quad \text{for}\ \ i=1,2,3.$$

This Lie algebra $(\mathbb{R}^9,[\cdot,\cdot])$ is isomorphic to the matrix Lie algebra $\bar{g}$, and the mapping σ, defined by (4.27), is a Lie isomorphism between the two Lie algebras.

We then define a bilinear form on $\mathbb{R}^9$ by

$$\langle a,b\rangle=a^TFb, \tag{4.31}$$

where F is a constant matrix. The symmetric property $\langle a,b\rangle=\langle b,a\rangle$ requires that

$$F^T=F. \tag{4.32}$$

Under this symmetric condition, the invariance property under the Lie bracket

$$\langle a,[b,c]\rangle=\langle[a,b],c\rangle$$

equivalently requires that

$$F(R(b))^T = -R(b)F, \quad b \in \mathbb{R}^9. \tag{4.33}$$

This matrix equation leads to a linear system of equations on the elements of F. Solving the resulting system yields

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\eta_3 & \alpha\beta\eta_3 \\ \eta_3 & \alpha\beta\eta_3 & \alpha\beta^2\eta_3 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{4.34}$$

where η_i, $1 \le i \le 3$, are arbitrary constants, and $\otimes$ is the Kronecker product.

Now, the corresponding bilinear form on the semi-direct sum $\bar{g}$ of Lie algebras is given by

$$\begin{aligned} \langle A,B\rangle &= \langle A,B\rangle_{\bar{g}} = \langle \sigma(A),\sigma(B)\rangle_{\mathbb{R}^9} = (a_1,\cdots,a_9)F(b_1,\cdots,b_9)^T \\ &= (2a_1b_1 + a_2b_3 + a_3b_2)\eta_1 + (2a_1b_4 + a_2b_6 + a_3b_5 + 2a_4b_1 + a_5b_3 + a_6b_2)\eta_2 \\ &\quad + (2a_1b_7 + a_2b_9 + a_3b_8 + 2\alpha a_4b_4 + 2\alpha\beta a_4b_7 + \alpha a_5b_6 + \alpha\beta a_5b_9 + \alpha a_6b_5 + \alpha\beta a_6b_8 \\ &\quad + 2a_7b_1 + 2\alpha\beta a_7b_4 + 2\alpha\beta^2 a_7b_7 + a_8b_3 + \alpha\beta a_8b_6 + \alpha\beta^2 a_8b_9 + a_9b_2 \\ &\quad + \alpha\beta^2 a_9b_8 + \alpha\beta a_9b_5)\eta_3, \end{aligned} \tag{4.35}$$

where $A = A(a_1,\cdots,a_9)$, $B = B(b_1,\cdots,b_9) \in \bar{g}$ are as defined in (4.28).

The bilinear form (4.35) is symmetric and invariant under the Lie bracket of the matrix Lie algebra:

$$\langle A,B\rangle = \langle B,A\rangle, \langle A,[B,C]\rangle = \langle [A,B],C\rangle,$$

where $A = A(a_1,\cdots,a_9)$, $B = B(b_1,\cdots,b_9)$, $C = C(c_1,\cdots,c_9) \in \bar{g}$ are as defined in (4.28). Obviously, this kind of bilinear forms is not of Killing type and is non-degenerate if and only if the determinant of the matrix F is non-zero:

$$\det(F) = 8\alpha^3(\eta_2\beta - \eta_3)^6\eta_3{}^3 \neq 0.$$

Therefore we can choose η_1, η_2, and η_3 such that $\det(F)$ is non-zero. Note that the two parameters α and β are arbitrary constants associated with the new class of matrix Lie algebras in (2.3), and they also should make $\det(F)$ non-zero to apply the variational identity.

Now we can compute that

$$\langle \bar{W}, \bar{U}_\lambda\rangle = -2\eta_1 a - 2\eta_2 e - 2\eta_3 e', \tag{4.36}$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}}\rangle = \begin{bmatrix} c\eta_1 + g\eta_2 + g'\eta_3 \\ b\eta_1 + f\eta_2 + f'\eta_3 \\ c\eta_2 + (\alpha g + \alpha\beta g')\eta_3 \\ b\eta_2 + (\alpha f + \alpha\beta f')\eta_3 \\ (c + \alpha\beta g + \alpha\beta^2 g')\eta_3 \\ (b + \alpha\beta f + \alpha\beta^2 f')\eta_3 \end{bmatrix},$$

and furthermore, we have

$$\gamma=-\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle\bar{W},\bar{W}\rangle|=0.$$

Thus, by the previous variational identity (4.26), we have

$$\frac{\delta}{\delta\bar{u}}\int\frac{2\eta_1 a_{m+1}+2\eta_2 e_{m+1}+2\eta_3 e'_{m+1}}{m}dx=\begin{bmatrix} c_m\eta_1+g_m\eta_2+g'_m\eta_3 \\ b_m\eta_1+f_m\eta_2+f'_m\eta_3 \\ c_m\eta_2+(\alpha g_m+\alpha\beta g'_m)\eta_3 \\ b_m\eta_2+(\alpha f_m+\alpha\beta f'_m)\eta_3 \\ (c_m+\alpha\beta g_m+\alpha\beta^2 g'_m)\eta_3 \\ (b_m+\alpha\beta f_m+\alpha\beta^2 f'_m)\eta_3 \end{bmatrix},\quad m\geq 1. \tag{4.37}$$

Consequently, we obtain the following Hamiltonian structures for the hierarchy of bi-integrable couplings (4.24):

$$\bar{u}_{t_m}=\bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \tag{4.38}$$

where the Hamiltonian functionals are

$$\bar{\mathcal{H}}_m=\int\frac{2\eta_1 a_{m+2}+2\eta_2 e_{m+2}+2\eta_3 e'_{m+2}}{m+1}dx,\quad m\geq 0, \tag{4.39}$$

and the Hamiltonian operator is

$$\bar{J}=\begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\eta_3 & \alpha\beta\eta_3 \\ \eta_3 & \alpha\beta\eta_3 & \alpha\beta^2\eta_3 \end{bmatrix}^{-1}\otimes J, \tag{4.40}$$

with matrix J being defined as in (4.9a).

4.4 Commutativity of symmetries and conserved functionals

The enlarged system (4.24) is also integrable in the sense that it possesses infinitely many commuting symmetries $\{\bar{K}_m\}_{m=0}^{\infty}$.

It is easy to check that

$$\bar{K}_m=\bar{\Phi}\bar{K}_{m-1},\quad m\geq 1, \tag{4.41}$$

where the hereditary recursion operator $\bar{\Phi}$ (see [23] for details) is defined by

$$\bar{\Phi}=\begin{bmatrix} \Phi & 0 & 0 \\ \Phi_1 & \Phi & 0 \\ \Phi_2 & \alpha\Phi_1+\alpha\beta\Phi_2 & \Phi+\alpha\beta\Phi_1+\alpha\beta^2\Phi_2 \end{bmatrix}=M^T(\Phi,\Phi_1,\Phi_2), \tag{4.42}$$

with M^T being the transpose of matrix M in (2.3), Φ being given as in (4.9a), and

$$\Phi_1=\begin{bmatrix} r\partial^{-1}q+p\partial^{-1}s & r\partial^{-1}p+p\partial^{-1}r \\ -s\partial^{-1}q-q\partial^{-1}s & -s\partial^{-1}p-q\partial^{-1}r \end{bmatrix}, \tag{4.43a}$$

$$\Phi_2=\begin{bmatrix} v\partial^{-1}q+\theta_1\partial^{-1}s+\theta_2\partial^{-1}w & v\partial^{-1}p+\theta_1\partial^{-1}r+\theta_2\partial^{-1}v \\ -w\partial^{-1}q-\theta_3\partial^{-1}s-\theta_4\partial^{-1}w & -w\partial^{-1}p-\theta_3\partial^{-1}r-\theta_4\partial^{-1}v \end{bmatrix}, \tag{4.43b}$$

in which

$$\begin{cases} \theta_1:=\alpha r+\alpha\beta v, & \theta_2:=p+\alpha\beta r+\alpha\beta^2 v, \\ \theta_3:=\alpha s+\alpha\beta w, & \theta_4:=q+\alpha\beta s+\alpha\beta^2 w. \end{cases} \tag{4.44}$$

It is obvious that $\bar{J}$ is skew symmetric and

$$\bar{J}\bar{\Phi}^*=\bar{\Phi}\bar{J}, \tag{4.45}$$

where $\bar{\Phi}^*$ denote the adjoint operator of $\bar{\Phi}$. Then we have $\bar{J}\bar{\Phi}^*$ is also skew symmetric. Furthermore, $\bar{J}$ and $\bar{M}=\bar{\Phi}\bar{J}$ form a Hamiltonian pair [4,22], and it follows that $\bar{\Phi}=\bar{M}\bar{J}^{-1}$ is hereditary operator (see [3,4]).

Consequently, there exist infinitely many commuting symmetries and conserved functionals:

$$[\bar{K}_m,\bar{K}_n]:=\bar{K}'_m(\bar{u})[\bar{K}_n]-\bar{K}'_n(\bar{u})[\bar{K}_m]=0, \qquad m,n\geq 0,$$
$$\{\bar{\mathcal{H}}_m,\bar{\mathcal{H}}_n\}:=\int\Big(\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}\Big)^T\bar{J}\frac{\delta\bar{\mathcal{H}}_n}{\delta\bar{u}}dx=0, \qquad m,n\geq 0.$$

It is easy to compute that for the n-th bi-integrable coupling system $\bar{u}_{t_n}=\bar{J}\delta\bar{\mathcal{H}}_n/\delta\bar{u}$,

$$\frac{d}{dt_n}\bar{\mathcal{H}}_m=\int\left(\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}\right)^T\bar{u}_{t_n}dx=\int\left(\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}\right)^T\bar{J}\frac{\delta\bar{\mathcal{H}}_n}{\delta\bar{u}}dx=0, \quad m\geq 0,$$

which implies that $\{\bar{\mathcal{H}}_m\}_{m\geq 0}$, are conserved, and each Hamiltonian coupling system has infinitely many commuting conserved functionals $\{\bar{\mathcal{H}}_m\}_{m\geq 0}$. Moreover, the resulting bi-integrable couplings possess the bi-Hamiltonian structure

$$\bar{u}_{t_m}=\bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}=\bar{M}\frac{\delta\bar{\mathcal{H}}_{m-1}}{\delta\bar{u}}, \quad m\geq 1.$$

5 Conclusions and remarks

The semi-direct sum of Lie algebras shows the mathematical structures for obtaining integrable couplings or multi-integrable couplings of given integrable systems.

The presented Lie algebras of 3×3 block matrices give a number of potential algebraic structures in finding bi-integrable couplings. Based on those new classes of matrix Lie algebras, and following similar schemes, we can generate bi-integrable couplings of other soliton hierarchies such as the KdV hierarchy and the Dirac hierarchy. We remark that another way to obtain bi-integrable couplings is to choose different types of submatrices U_i in the spectral matrix $\bar{U}$ defined in (3.2).

Among all 10 classes of non-semisimple Lie algebras presented in this paper, $\text{Class}_{\mathbf{5}}$ and $\text{Class}_{\mathbf{8}}$ are among the most interesting ones. They have more than one parameter, and taking special reductions of the parameters, we can obtain interesting classes of Lie algebras of block matrices, so as to obtain bi-integrable couplings

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u,u_1), \\ u_{2,t} = S_2(u,u_1,u_2). \end{cases}$$

Note that we don't keep all the parameters, otherwise, the subsystems $u_t = K$, $u_{1,t} = S_1$, and $u_{2,t} = S_2$ might be independent of each other, so the enlarged integrable system will be trivial bi-integrable couplings.

Some other classes, for example, $\text{Class}_{\mathbf{1}}$, $\text{Class}_{\mathbf{6}}$, and $\text{Class}_{\mathbf{7}}$, might not produce Hamiltonian structures. One example of $\text{Class}_{\mathbf{6}}$ has already been studied in [17], and the Lax pair is in the form of

$$\bar{U} = \begin{bmatrix} U & U_1 & U_2 \\ 0 & U & 0 \\ 0 & 0 & U \end{bmatrix}, \qquad \bar{V} = \begin{bmatrix} V & V_1 & V_2 \\ 0 & V & 0 \\ 0 & 0 & V \end{bmatrix}.$$

However, it is difficult to determine whether integrable couplings generated by the above type of non-semisimple Lie algebras possess Hamiltonian structures or not, since any bilinear form satisfying the symmetric and invariant conditions of the variational identity is degenerate. This is the case also for $\text{Class}_{\mathbf{7}}$. Our question is: for those non-semisimple matrix Lie algebras of 3×3 bock matrices, can we reduce restrictions on bilinear forms in the variational identity to find Hamiltonian structures?

Moreover, by using the Kronecker product of matrices [18], we can get new Lax pairs and new zero curvature representations for bi-integrable couplings.

In conclusion, the subject of bi-integrable couplings, initiated more than one decade ago, is rather interesting. We are going to explore more different classes of matrix Lie algebras in the future.

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