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A Priori Error Estimates of Crank-Nicolson Finite Volume Element Method for Parabolic Optimal Control Problems

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Abstract. In this paper, the Crank-Nicolson linear finite volume element method is applied to solve the distributed optimal control problems governed by a parabolic equation. The optimal convergent order $\mathcal{O}(h^2+k^2)$ is obtained for the numerical solution in a discrete L^2-norm. A numerical experiment is presented to test the theoretical result.

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Key words: Variational discretization, parabolic optimal control problems, finite volume element method, distributed control, Crank-Nicolson.

1 Introduction

The optimal control problems introduced in [13] are playing an increasingly important role in science and engineering. They have various applications in the operation of physical, social, and economic processes. Many numerical methods, such as finite element method, mixed finite element method, spectral method, streamline finite element method etc., have been applied to approximate the solutions of these problems (see, e.g., [3–8, 10, 14]).

In [16], to our best knowledge, the authors first use the finite volume element method to obtain the numerical solution for an optimal control problem associate with a parabolic

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equation by using *optimize-then-discretize* approach and the variational discretization technique (proposed in [12]). Also, the authors derive some error estimates for the semi-discrete solution and fully-discrete approximation. For the fully-discrete approximation, the convergent order is $\mathcal{O}(h^2+k)$ there. Here we develop the Crank-Nicolson linear finite volume element method for solving the parabolic optimal control problems and get the optimal order $\mathcal{O}(h^2+k^2)$.

In this paper, we consider the following optimal control problems: Find y, u such that

$$\min_{u\in U_{ad}} \frac{1}{2}\int_0^T \big(\|y(\tau,x)-y_d(\tau,x)\|^2_{L^2(\Omega)}+\alpha\|u(\tau,x)\|^2_{L^2(\Omega)}\big)d\tau, \tag{1.1a}$$

$$y_t(t,x)-\nabla\cdot(A\nabla y(t,x))=Bu(t,x)+f(t,x), \quad t\in J, \quad x\in\Omega, \tag{1.1b}$$

$$y(t,x)=0, \quad t\in J, \quad x\in\Gamma, \quad y(0,x)=y_0, \quad x\in\Omega, \tag{1.1c}$$

where

$$\nabla\cdot(A\nabla y)=\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big),$$

$\Omega\subset R^2$ is a bounded convex polygon domain and Γ is the boundary of Ω, α is a positive number, $f(t,\cdot),y_d(t,\cdot)\in L^2(\Omega)$ or $H^1(\Omega)$, $J=(0,T]$, $A=(a_{i,j})_{2\times 2}$ is a symmetric, smooth enough and uniformly positive definite matrix in Ω, $B: L^2(J;L^2(\Omega))\rightarrow L^2(J;L^2(\Omega))$ is a continuous linear operator, $y_0(x)=0$, $x\in\Gamma$, U_{ad} is a set defined by

$$U_{ad}=\big\{u:\ u\in L^2(J;L^2(\Omega)),\ a\le u(t,x)\le b,\ a.e. \text{ in } \Omega,\ t\in J,\ a,b\in \mathrm{R}\big\}.$$

A semi-discrete optimal system is carried out in [16] and the existence and uniqueness of the solution for the system is proved there. Here we use the Crank-Nicolson scheme to discretize the semi-discrete optimal system and obtain the optimal convergent order $\mathcal{O}(h^2+k^2)$.

The remainder of this paper is organized as follows. In Section 2, we present some notations. In Section 3, we present the Crank-Nicolson linear finite volume element method for the optimal control problems. In Section 4, we first show some lemmas and then analyze the error estimate between the exact solution and the Crank-Nicolson linear finite volume element approximation. And in Section 5, a numerical example is presented to test the theoretical results.

Throughout this paper, the constant C denotes different positive constant at each occurrence, which is independent of the mesh size h and the time step k.

2 Notations

We use the standard notations $W^{m,p}(\Omega)$ for Sobolev spaces and their associated norms $\|v\|_{m,p}$ (see, e.g., [1]) in this paper. To simplify the notations, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and drop the index $p=2$ and Ω whenever possible, i.e.,

$$\|u\|_{m,2,\Omega}=\|u\|_{m,2}=\|u\|_m, \quad \|u\|_0=\|u\|.$$

Set $H_0^1(\Omega)=\{v\in H^1: v|_{\partial\Omega}=0\}$. As usual, we use $(\cdot,\cdot)$ to denote the $L^2(\Omega)$-inner product.

Let $0=t_0<t_1<\cdots<t_{M-1}<t_M=T$ be a subdivision of J with time step $k_i=t_i-t_{i-1}$, $i=1,2,\cdots,M$. Let $v^i=v(t_i,x)$, and $\partial v^i=(v^i-v^{i-1})/k_i$. We define a discrete time-dependent norm

$$|||v|||=\Big(\sum_{i=1}^{M}k_i\|v^{i-\frac{1}{2}}\|^2\Big)^{\frac{1}{2}}.$$

For a convex polygonal domain Ω, we consider a quasi-uniform triangulation $\mathcal{T}_h$ consisting of closed triangle elements K such that $\bar{\Omega}=\cup_{K\in\mathcal{T}_h}K$. We use N_h to denote the set of all nodes or vertices of $\mathcal{T}_h$. To define the dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$, we divide each $K\in\mathcal{T}_h$ into three quadrilaterals by connecting the barycenter C_K of K with line segments to the midpoints of edges of K. The control volume V_i consists of the quadrilaterals sharing the same vertex z_i as is shown in Fig. 1. The dual partition $\mathcal{T}_h^*$ consists of the union of the control volume V_i. Let $h=\max\{h_K\}$, where h_K is the diameter of the triangle K. As is shown in [11], the dual partition $\mathcal{T}_h^*$ is also quasi-uniform.

We define the finite dimensional space V_h (i.e., trial space) associated with $\mathcal{T}_h$ for the trial functions by

$$V_h=\{v:\ v\in C(\Omega),\ \ v|_K\in P_1(K),\ \ \forall K\in\mathcal{T}_h,\ \ v|_\Gamma=0\}$$

and define the finite dimensional space Q_h (i.e., test space) associated with the dual partition $\mathcal{T}_h^*$ for the test functions by

$$Q_h=\{q:\ q\in L^2(\Omega),\ \ q|_V\in P_0(V),\ \ \forall\ V\in\mathcal{T}_h^*;\ \ q|_{V_z}=0,\ \ z\in\Gamma\},$$

where $P_l(K)$ or $P_l(V)$ consists of all the polynomials with degree less than or equal to l defined on K or V.

To connect the trial space and test space, we define a transfer operator $I_h: V_h\to Q_h$ as

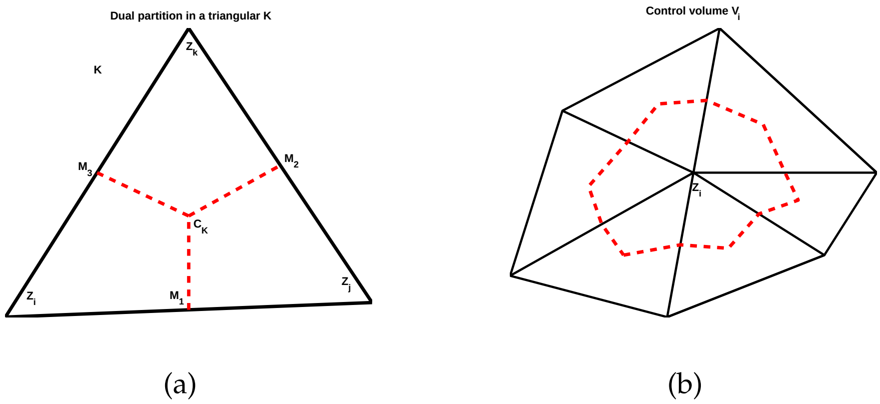

Figure 1: The dual partition of a triangular K on the left hand side and a control volume V_i on the right hand side.

follows:

$$I_h v_h = \sum_{z_i \in N_h} v_h(z_i)\chi_i, \quad I_h v_h|_{V_i} = v_h(z_i), \quad \forall V_i \in \mathcal{T}_h^*,$$

where χ_i is the characteristic function of V_i.

Let

$$a(w,v) = \int_\Omega A\nabla w \cdot \nabla v dx.$$

As is defined in [2, (1.10)], we define the standard Ritz projection $R_h : H_0^1 \to V_h$ by

$$a(R_h u, \chi) = a(u, \chi), \quad \forall \chi \in V_h. \tag{2.1}$$

3 Crank-Nicolson finite volume scheme

In this section, we will use the *optimize-then-discretize* approach to obtain the semi-discrete finite volume element scheme for the parabolic optimal control problems. Then we present the Crank-Nicolson fully-discrete finite volume element scheme.

As is seen in [13], the necessary and sufficient optimal condition (system) of (1.1a)-(1.1c) consists of the state equation, a costate equation and a variational inequality, i.e., find $y(t,\cdot)$, $p(t,\cdot) \in H_0^1(\Omega)$ and $u(t,\cdot) \in U_{ad}$ such that

$$(y_t, w) + (A\nabla y, \nabla w) = (Bu + f, w), \qquad \forall\, w \in H_0^1(\Omega), \quad y(0,x) = y_0(x), \tag{3.1a}$$

$$-(p_t, q) + (A\nabla p, \nabla q) = (y - y_d, q), \qquad \forall\, q \in H_0^1(\Omega), \quad p(T,x) = 0, \tag{3.1b}$$

$$\int_0^T (\alpha u + B^* p, v - u) d\tau \geq 0, \qquad \forall\, v \in U_{ad}. \tag{3.1c}$$

If $y(t,\cdot) \in H_0^1(\Omega) \cap C^2(\Omega)$ and $p(t,\cdot) \in H_0^1(\Omega) \cap C^2(\Omega)$, then the optimal system (3.1a)-(3.1c) can be written by

$$y_t - \nabla\cdot(A\nabla y) = Bu + f, \quad t \in J, \quad x \in \Omega, \tag{3.2a}$$

$$y(t,x) = 0, \quad t \in J, \quad x \in \Gamma, \quad y(0,x) = y_0(x), \quad x \in \Omega;$$

$$-p_t - \nabla\cdot(A\nabla p) = y - y_d, \quad t \in J, \quad x \in \Omega, \tag{3.2b}$$

$$p(t,x) = 0, \quad t \in J, \quad x \in \Gamma, \quad p(T,x) = 0, \quad x \in \Omega;$$

$$\int_0^T (\alpha u + B^* p, v - u) d\tau \geq 0, \quad \forall v \in U_{ad}. \tag{3.2c}$$

We use the piecewise linear finite volume element method to discretized the state and costate equations directly. Then the continuous optimal system (3.2a)-(3.2c) can be approximated by: Find $(y_h(t,\cdot),p_h(t,\cdot),u_h(t,\cdot))\in V_h\times V_h\times U_{ad}$ such that

$$
\begin{aligned}
&(y_{h,t},I_hw_h)+a_h(y_h,I_hw_h)=(Bu_h+f,I_hw_h), && \forall w_h\in V_h, && \text{(3.3a)}\\
&y_h(0,x)=R_hy_0(x), && x\in\Omega;\\
&-(p_{h,t},I_hq_h)+a_h(p_h,I_hq_h)=(y_h-y_d,I_hq_h), && \forall q_h\in V_h, && \text{(3.3b)}\\
&p_h(T,x)=0, && x\in\Omega;\\
&\int_0^T(\alpha u_h+B^*p_h,v-u_h)d\tau\geq 0, && \forall v\in U_{ad}, && \text{(3.3c)}
\end{aligned}
$$

where

$$a_h(\phi,I_h\psi)=-\sum_{z_i\in N_h}\psi(z_i)\int_{\partial V_i}A\nabla\phi\cdot\mathbf{n}ds.$$

The variational discretization technique is used here for the variational inequality.

The existence and uniqueness of the solution for the system (3.3a)-(3.3c) have been proved in [16]. Here we present the Crank-Nicolson linear finite volume element method (**CN-FVEM**) for the former system. Find $(y_h^i,p_h^i,u_h^{i-1/2})\in V_h\times V_h\times U_{ad}$ such that for all $w_h\in V_h$, $q_h\in V_h$, $v\in U_{ad}$

$$
\begin{aligned}
&(\partial y_h^i,I_hw_h)+a_h\Big(\frac{y_h^i+y_h^{i-1}}{2},I_hw_h\Big)=(Bu_h^{i-\frac12}+f^{i-\frac12},I_hw_h), && \text{(3.4a)}\\
&y_h^0(x)=R_hy_0(x),\quad x\in\Omega;\quad i=1,\cdots,M,\\
&-(\partial p_h^i,I_hq_h)+a_h\Big(\frac{p_h^i+p_h^{i-1}}{2},I_hq_h\Big)=\Big(\frac{y_h^i+y_h^{i-1}}{2}-y_d^{i-\frac12},I_hq_h\Big), && \text{(3.4b)}\\
&p_h^M(x)=0,\quad x\in\Omega;\quad i=M,\cdots,1,\\
&\Big(\alpha u_h^{i-\frac12}+B^*\Big(\frac{p_h^i+p_h^{i-1}}{2}\Big),v-u_h^{i-\frac12}\Big)\geq 0,\quad i=1,\cdots,M. && \text{(3.4c)}
\end{aligned}
$$

Let $u_h^M=\max(a,\min(b,0))$ (The rationality can be seen (4.20) and (4.21)). Let $\overline{U}_h$ be the linear interpolation of $u_h^{1/2},u_h^{3/2},\cdots,u_h^{M-1/2},u_h^M$ such that $\overline{U}_h((t_i+t_{i-1})/2,\cdot)=u_h^{i-1/2}$, $i=1,2,\cdots,M$, $U_h=\max(a,\min(b,\overline{U}_h))$.

4 Error analysis

In this section, we first present some known results. Then using these known results, we analyse the error of the Crank-Nicolson linear finite volume element approximation for a parabolic problem. After that we prove an auxiliary lemma and derive some error estimates for the Crank-Nicolson fully-discrete finite volume element approximation.

4.1 Some known results

For the operator I_h, it is well known that there exists a positive constant C such that for all $v \in V_h$

$$\|v - I_h v\| \leq Ch\|v\|_1. \tag{4.1}$$

For the projection R_h defined in (2.1), it has the property that (see, e.g., [2, (4.2)])

$$\|R_h u - u\| \leq Ch^2 \|u\|_2. \tag{4.2}$$

Let $\varepsilon_h(x,y) = (x,y) - (x, I_h y)$, $\varepsilon_a(x,y) = a(x,y) - a_h(x, I_h y)$ and $\pi(x,y) = a_h(x, I_h y) - a_h(y, I_h x)$. it is well known (see, e.g., [2, Lemma4.1], [9, Lemma2.4]) that for all $y \in V_h$:

$$|\varepsilon_h(x,y)| \leq Ch^2 \|x\|_1 \|y\|_1, \qquad x \in H^1, \tag{4.3a}$$

$$|\varepsilon_a(x,y)| \leq Ch \|x\|_1 \|y\|_1, \qquad x \in V_h, \tag{4.3b}$$

$$|\varepsilon_a(R_h x, y)| \leq Ch^2 \|x\|_2 \|y\|_1, \qquad x \in H^2 \cap H_0^1, \tag{4.3c}$$

$$|\pi(x,y)| \leq Ch \|x\|_1 \|y\|_1, \qquad x \in V_h. \tag{4.3d}$$

4.2 Error analysis of Crank-Nicolson finite volume element method

We consider the following problem

$$\begin{cases} w_t(t,x) - \nabla \cdot (A \nabla w(t,x)) = f(t,x), \quad t \in J, \quad x \in \Omega, \\ w(t,x) = 0, \quad t \in J, \quad x \in \Gamma, \quad w(0,x) = w_0, \quad x \in \Omega, \end{cases} \tag{4.4}$$

where A, J, Ω are as described as in (1.1a)-(1.1c). The Crank-Nicolson linear finite volume element method for the problem (4.4) is to find $W^i \in V_h$ such that

$$\begin{cases} (\partial W^i, I_h \chi) + a_h\Big(\dfrac{W^i + W^{i-1}}{2}, I_h \chi\Big) = (f^{i-1/2}, I_h \chi), \quad \forall \chi \in V_h, \quad i = 1, \cdots, M, \\ W^0 = R_h w_0. \end{cases} \tag{4.5}$$

For the Crank-Nicolson finite volume element method, we have the following results.

Theorem 4.1. *Assume that W^n, $w^n = w(t^n, x)$, $n = 1, 2, \cdots, M$ are the solutions of* (4.5) *and* (4.4), *respectively. Then the following result holds:*

$$\|W^n - w^n\| \leq C(h^2 + k^2). \tag{4.6}$$

Proof. We can easily get from (4.4)

$$\begin{cases} (w_t(t_{i-1/2}), I_h \chi) + a_h(w(t_{i-1/2}), I_h \chi) = (f^{i-1/2}, I_h \chi), \quad \forall \chi \in V_h, \quad i = 1, \cdots, M, \\ w^0 = w_0. \end{cases} \tag{4.7}$$

Subtracting (4.7) from (4.5), we can obtain

$$(\partial W^i - w_t(t_{i-1/2}), I_h\chi) + a_h\Big(\frac{W^i+W^{i-1}}{2} - w(t_{i-1/2}), I_h\chi\Big) = 0, \quad \forall \chi \in V_h, \quad i=1,\cdots,M.$$

Let

$$\theta^i = W^i - R_h w(t_i), \quad \rho^i = R_h w(t_i) - w(t_i).$$

Then we have

$$\begin{aligned}
&(\partial\theta^i, I_h\chi) + a_h\Big(\frac{\theta^i+\theta^{i-1}}{2}, I_h\chi\Big)\\
=&(w_t(t_{i-1/2}) - \partial w(t_i), I_h\chi) + (\partial w(t_i) - R_h \partial w(t_i), I_h\chi)\\
&+a_h\Big(w(t_{i-1/2}) - R_h\Big(\frac{w(t_i)+w(t_{i-1})}{2}\Big), I_h\chi\Big)\\
=&I_1+I_2+I_3.
\end{aligned} \tag{4.8}$$

Using the equivalent property of $(\cdot,\cdot)$, $(\cdot,I_h\cdot)$, $(I_h\cdot,I_h\cdot)$, we can derive

$$\begin{aligned}
I_1 \le& C\|w_t(t_{i-1/2}) - \partial w(t_i)\|\,\|\chi\| \le Ck^{-1}\cdot k^2\int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|ds\cdot\|\chi\|\\
\le& Ck^{\frac{3}{2}}\Big(\int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|^2 ds\Big)^{\frac{1}{2}}\cdot\|\chi\| \le C(\epsilon)k^3\int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|^2 ds + \epsilon\|\chi\|^2.
\end{aligned} \tag{4.9}$$

The property (4.2) of R_h implies

$$\begin{aligned}
I_2 \le& Ch^2\|\partial w(t_i)\|_2\cdot\|\chi\| \le Ch^2k^{-1}\int_{t_{i-1}}^{t_i}\|w_t(s)\|_2 ds\cdot\|\chi\|\\
\le& Ch^2k^{-1/2}\Big(\int_{t_{i-1}}^{t_i}\|w_t(s)\|_2^2 ds\Big)^{1/2}\cdot\|\chi\| \le C(\epsilon)h^4k^{-1}\int_{t_{i-1}}^{t_i}\|w_t(s)\|_2^2 ds + \epsilon\|\chi\|^2.
\end{aligned} \tag{4.10}$$

For the third term I_3, we have

$$\begin{aligned}
I_3 =& \Big(f^{i-1/2} - \frac{f^i+f^{i-1}}{2}, I_h\chi\Big) - \Big(w_t^{i-1/2} - \frac{w_t^i + w_t^{i-1}}{2}, I_h\chi\Big) + \varepsilon_a\Big(R_h\Big(\frac{w^i+w^{i-1}}{2}\Big), \chi\Big)\\
\le& Ck\Big(\int_{t_{i-1}}^{t_i}\|f_{tt}(s)\|ds + \int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|ds\Big)\|\chi\| + Ch^2\Big\|\frac{w^i+w^{i-1}}{2}\Big\|_2\|\chi\|_1\\
\le& Ck^{3/2}\Big[\Big(\int_{t_{i-1}}^{t_i}\|f_{tt}(s)\|^2 ds\Big)^{1/2} + \Big(\int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|^2 ds\Big)^{1/2}\Big]\|\chi\| + Ch^2\Big\|\frac{w^i+w^{i-1}}{2}\Big\|_2\|\chi\|_1\\
\le& C(\epsilon)k^3\Big[\int_{t_{i-1}}^{t_i}\|f_{tt}(s)\|^2 ds + \int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|^2 ds\Big] + C(\epsilon)h^4\Big\|\frac{w^i+w^{i-1}}{2}\Big\|_2^2 + \epsilon\|\chi\|_1^2.
\end{aligned} \tag{4.11}$$

Using (4.8)-(4.11), selecting appropriate value for ϵ and choosing $\chi=(\theta^i+\theta^{i-1})/2$, we can obtain

$$\begin{aligned}
&\frac{1}{2k_i}[(\theta^i,I_h\theta^i)-(\theta^{i-1},I_h\theta^{i-1})]+a_h\Big(\frac{\theta^i+\theta^{i-1}}{2},I_h\Big(\frac{\theta^i+\theta^{i-1}}{2}\Big)\Big)\\
=&\Big(\partial\theta^i,I_h\Big(\frac{\theta^i+\theta^{i-1}}{2}\Big)\Big)+a_h\Big(\frac{\theta^i+\theta^{i-1}}{2},I_h\Big(\frac{\theta^i+\theta^{i-1}}{2}\Big)\Big)\\
\le& Ck^3\Big[\int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|^2ds+\int_{t_{i-1}}^{t_i}\|f_{tt}(s)\|^2ds\Big]+Ch^4\Big[k^{-1}\int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|_2^2ds+\Big\|\frac{w^i+w^{i-1}}{2}\Big\|_2^2\Big]\\
&+C_0\Big\|\frac{\theta^i+\theta^{i-1}}{2}\Big\|_1^2,
\end{aligned}$$

where C_0 is the C_0 in "$a_h(v_h,I_hv_h)\ge C_0\|v_h\|_1^2$", which can be found in [2, (3.6)]. Removing the terms of the former inequality and noticing the equivalent property of $(\cdot,\cdot)$, $(\cdot,I_h\cdot)$, $(I_h\cdot,I_h\cdot)$, we can get

$$\begin{aligned}
\|\theta^i\|^2\le\|\theta^{i-1}\|^2+&Ck^4\Big[\int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|^2ds+\int_{t_{i-1}}^{t_i}\|f_{tt}(s)\|^2ds\Big]\\
&+Ch^4\Big[\int_{t_{i-1}}^{t_i}\|w_{ttt}(s)\|_2^2ds+k_i\Big\|\frac{w^i+w^{i-1}}{2}\Big\|_2^2\Big].
\end{aligned}$$

Summing i from 1 to n and noticing that $\theta^0=0$, we have

$$\begin{aligned}
\|\theta^n\|^2\le& Ck^4\Big[\int_0^{t_n}\|w_{ttt}(s)\|^2ds+\int_0^{t_n}\|f_{tt}(s)\|^2ds\Big]\\
&+Ch^4\Big[\int_0^{t_n}\|w_{ttt}(s)\|_2^2ds+\int_0^{t_n}\|w(s)\|_2^2ds\Big].
\end{aligned}\tag{4.12}$$

Using (4.12), (4.2) and the triangle inequality, we can get (4.6) easily. □

Remark 4.1. For the Crank-Nicolson linear finite volume element method of the problem

$$\begin{cases}
-w_t(t,x)-\nabla\cdot(A\nabla w(t,x))=f(t,x), \quad t\in J, \quad x\in\Omega,\\
w(t,x)=0, \quad t\in J, \quad x\in\Gamma, \quad w(T,x)=0, \quad x\in\Omega,
\end{cases}$$

it has the same convergent order.

4.3 An auxiliary lemma

To derive the fully discrete error analysis, let $y_h^i(u)$ be the solution of

$$\begin{cases}
(\partial y_h^i(u),I_hw_h)+a_h\Big(\frac{y_h^i(u)+y_h^{i-1}(u)}{2},I_hw_h\Big)=(Bu^{i-\frac12}+f^{i-\frac12},I_hw_h),\\
\qquad\qquad\qquad\qquad\qquad\qquad \forall w_h\in V_h, \quad i=1,\cdots,M,\\
y_h^0(u)(x)=R_hy_0, \quad x\in\Omega,
\end{cases}\tag{4.13}$$

and $p_h^i(y)$ be the solution of

$$\begin{cases} -(\partial p_h^i(y),I_hq_h)+a_h\Big(\dfrac{p_h^i(y)+p_h^{i-1}(y)}{2},I_hq_h\Big)=(y^{i-\frac{1}{2}}-y_d^{i-\frac{1}{2}},I_hq_h), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall q_h\in V_h, \quad i=M,\cdots,1, \\ p_h^M(y)(x)=0, \quad x\in\Omega. \end{cases} \tag{4.14}$$

Let $u_h=(u_h^0,u_h^1,\cdots,u_h^M)$, $y_h=(y_h^0,y_h^1,\cdots,y_h^M)$ and $p_h=(p_h^0,p_h^1,\cdots,p_h^M)$. For $y_h^i(u)$, $p_h^i(y)$, noticing that $y_h^i=y_h^i(u_h)$, $p_h^i=p_h^i(y_h)$, we have the following lemma.

Lemma 4.1. *Assume that $y_h^n(u),p_h^n(y),n=0,1,2,\cdots,M$ are the solutions of* (4.13) *and* (4.14), *respectively. Then the following results hold:*

$$\|y_h^n(u)-y_h^n\|_1\le C|\|u-U_h\||, \tag{4.15a}$$
$$\|p_h^n(y)-p_h^n\|_1\le C|\|u-U_h\||+C(h^2+k^2). \tag{4.15b}$$

Proof. Let $\eta^i=y_h^i(u)-y_h^i$. Subtracting (4.13) from (3.4a), we have

$$(\partial\eta^i,I_hw_h)+a_h\Big(\frac{\eta^i+\eta^{i-1}}{2},I_hw_h\Big)=\big(B(u^{i-1/2}-u_h^{i-1/2}),I_hw_h\big), \quad \forall w_h\in V_h.$$

Take $w_h=\partial\eta^i$. We have

$$(\partial\eta^i,I_h\partial\eta^i)+a\Big(\frac{\eta^i+\eta^{i-1}}{2},\partial\eta^i\Big)=\varepsilon_a\Big(\frac{\eta^i+\eta^{i-1}}{2},\partial\eta^i\Big)+(B(u^{i-1/2}-u_h^{i-1/2}),I_h\partial\eta^i).$$

Noticing that

$$a\Big(\frac{\eta^i+\eta^{i-1}}{2},\partial\eta^i\Big)=\frac{1}{2k_i}\big[a(\eta^i,\eta^i)-a(\eta^{i-1},\eta^{i-1})\big],$$

we obtain

$$\begin{aligned}&(\partial\eta^i,I_h\partial\eta^i)+\frac{1}{2k_i}\big[a(\eta^i,\eta^i)-a(\eta^{i-1},\eta^{i-1})\big]\\ =&\varepsilon_a\Big(\frac{\eta^i+\eta^{i-1}}{2},\partial\eta^i\Big)+\big(B(u^{i-1/2}-u_h^{i-1/2}),I_h\partial\eta^i\big).\end{aligned}$$

The inverse estimate and (4.3b) imply that

$$\begin{aligned}\varepsilon_a\Big(\frac{\eta^i+\eta^{i-1}}{2},\partial\eta^i\Big)\le& Ch\Big\|\frac{\eta^i+\eta^{i-1}}{2}\Big\|_1\|\partial\eta^i\|_1\le C\Big\|\frac{\eta^i+\eta^{i-1}}{2}\Big\|_1\|\partial\eta^i\|\\ \le& C(\epsilon)\Big\|\frac{\eta^i+\eta^{i-1}}{2}\Big\|_1^2+\epsilon\|\partial\eta^i\|^2\le C(\epsilon)(\|\eta^i\|_1^2+\|\eta^{i-1}\|_1^2)+\epsilon\|\partial\eta^i\|^2.\end{aligned}$$

The property of continuity for B implies that

$$\big(B(u^{i-1/2}-u_h^{i-1/2}),I_h\partial\eta^i\big)\le C(\epsilon)\|u^{i-1/2}-u_h^{i-1/2}\|^2+\epsilon\ (I_h\partial\eta^i,I_h\partial\eta^i).$$

Using the equivalent properties of $(\cdot,\cdot)$, $(\cdot,I_h\cdot)$ and $(I_h\cdot,I_h\cdot)$, selecting an appropriate value for ϵ, we have that

$$a(\eta^i,\eta^i)\le a(\eta^{i-1},\eta^{i-1})+Ck_i(\|\eta^i\|_1^2+\|\eta^{i-1}\|_1^2)+Ck_i\|u^{i-1/2}-u_h^{i-1/2}\|^2.$$

Summing i from 1 to n and noticing $\eta^0=0$, we have that

$$\|\eta^n\|_1^2\le\sum_{i=1}^n Ck_i\|\eta^i\|_1^2+\sum_{i=1}^n Ck_i\|u^{i-1/2}-u_h^{i-1/2}\|^2,$$

where the coercive property of $a(\cdot,\cdot)$ is used. Using the discrete Gronwall's lemma, we have

$$\|\eta^n\|_1=\|y_h^n(u)-y_h^n\|_1\le C|||u-U_h|||.$$

Let $\tau^i=p_h^i(y)-p_h^i$. Similarly, we can obtain

$$a(\tau^{i-1},\tau^{i-1})\le a(\tau^i,\tau^i)+Ck_i(\|\tau^i\|_1^2+\|\tau^{i-1}\|_1^2)+Ck_i\Big\|\frac{y_h^i+y_h^{i-1}}{2}-y^{i-1/2}\Big\|^2.$$

Summing i from $n+1$ to M and noticing $\tau^M=0$, we have that

$$\|\tau^n\|_1^2\le\sum_{i=n}^M Ck_i\|\tau^i\|_1^2+\sum_{i=1}^M Ck_i\Big\|\frac{y_h^i+y_h^{i-1}}{2}-y^{i-1/2}\Big\|^2.$$

Using the discrete Gronwall's lemma, we can achieve

$$\|\tau^n\|_1^2\le C\sum_{i=1}^M k_i\Big\|\frac{y_h^i+y_h^{i-1}}{2}-y^{i-1/2}\Big\|^2.$$

Moreover

$$\begin{aligned}\Big\|\frac{y_h^i+y_h^{i-1}}{2}-y^{i-1/2}\Big\|\le&\Big\|\frac{y_h^i+y_h^{i-1}}{2}-\frac{y_h^i(u)+y_h^{i-1}(u)}{2}\Big\|+\Big\|\frac{y_h^i(u)+y_h^{i-1}(u)}{2}-\frac{y^i+y^{i-1}}{2}\Big\|\\&+\Big\|\frac{y^i+y^{i-1}}{2}-y^{i-1/2}\Big\|.\end{aligned}\tag{4.16}$$

Therefore we can get (4.15b) from (4.15a), (4.16) and Theorem 4.1. □

4.4 Error analysis of CN-FVEM for parabolic optimal control problems

Using Lemma 4.1, we can get the error estimate for U_h in the discrete norm $|||\cdot|||$.

Theorem 4.2. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of problems* (3.1a)-(3.1c) *and* (3.4a)-(3.4c), *respectively. Then there exists an* $h_0>0$ *such that for all* $0<h\le h_0$

$$|||u-U_h|||\le C(h^2+k^2).\tag{4.17}$$

Proof. Noticing (3.1c) and (3.4c), we have

$$
\begin{aligned}
\alpha|||u-U_h|||^2=&\sum_{i=1}^M k_i\alpha\big(u^{i-1/2}-u_h^{i-1/2},u^{i-1/2}-u_h^{i-1/2}\big)\\
=&-\sum_{i=1}^M k_i\big(\alpha u^{i-1/2}+B^*p^{i-1/2},u_h^{i-1/2}-u^{i-1/2}\big)\\
&-\sum_{i=1}^M k_i\Big(\alpha u_h^{i-1/2}+B^*\Big(\frac{p_h^i+p_h^{i-1}}{2}\Big),u^{i-1/2}-u_h^{i-1/2}\Big)\\
&+\sum_{i=1}^M k_i\Big(B^*\Big(\frac{p_h^i+p_h^{i-1}}{2}-p^{i-1/2}\Big),u^{i-1/2}-u_h^{i-1/2}\Big)\\
\le&\sum_{i=1}^M k_i\Big(B^*\Big(\frac{p_h^i+p_h^{i-1}}{2}-p^{i-1/2}\Big),u^{i-1/2}-u_h^{i-1/2}\Big)\\
=&\sum_{i=1}^M k_i\Big(\frac{p^i+p^{i-1}}{2}-p^{i-1/2},B\big(u^{i-1/2}-u_h^{i-1/2}\big)\Big)\\
&+\sum_{i=1}^M k_i\Big(\frac{p_h^i+p_h^{i-1}}{2}-\frac{p^i+p^{i-1}}{2},B\big(u^{i-1/2}-u_h^{i-1/2}\big)\Big)\\
\doteq&T_1+T_2.
\end{aligned}
$$

Using the Cauchy inequality, we obtain that

$$
\begin{aligned}
T_1=&\sum_{i=1}^M k_i\Big(\frac{p^i+p^{i-1}}{2}-p^{i-1/2},B\big(u^{i-1/2}-u_h^{i-1/2}\big)\Big)\\
\le&C\sum_{i=1}^M k_i\Big\|\frac{p^i+p^{i-1}}{2}-p^{i-1/2}\Big\|\|u^{i-1/2}-u_h^{i-1/2}\|\\
\le&C\sum_{i=1}^M k_i^2\int_{t_{i-1}}^{t_i}\|p_{tt}(s)\|ds\|u^{i-1/2}-u_h^{i-1/2}\|\\
\le&Ck^2\Big(\int_0^T\|p_{tt}(s)\|^2ds\Big)^{1/2}|||u-U_h|||.
\end{aligned}
$$

For the second term T_2, we can derive

$$
\begin{aligned}
T_2=&\sum_{i=1}^M k_i\Big(\frac{p_h^i+p_h^{i-1}}{2}-\frac{p^i+p^{i-1}}{2},B(u^{i-1/2}-u_h^{i-1/2})\Big)\\
=&\sum_{i=1}^M k_i\Big(\frac{p_h^i(y)+p_h^{i-1}(y)}{2}-\frac{p^i+p^{i-1}}{2},B(u^{i-1/2}-u_h^{i-1/2})\Big)\\
&+\sum_{i=1}^M k_i\Big(\frac{p_h^i+p_h^{i-1}}{2}-\frac{p_h^i(y)+p_h^{i-1}(y)}{2},B(u^{i-1/2}-u_h^{i-1/2})\Big)\\
\doteq&J_1+J_2.
\end{aligned}
$$

Using the Cauchy inequality and Remark 4.1, we have

$$\begin{aligned} J_1 \le & C\sum_{i=1}^{M} k_i\big(p_h^i(y)-p^i, B(u^{i-1/2}-u_h^{i-1/2})\big) \\ \le & C\sum_{i=1}^{M} k_i \ \|p_h^i(y)-p^i\| \ \|u^{i-1/2}-u_h^{i-1/2}\| \\ = & C\sum_{i=1}^{M} k_i^{1/2} \ \|p_h^i(y)-p^i\| \ k_i^{1/2}\|u^{i-1/2}-u_h^{i-1/2}\| \\ \le & C(h^2+k^2)|\|u-U_h\||. \end{aligned}$$

Let

$$\gamma^i = \frac{(p_h^i+p_h^{i-1})}{2} - \frac{(p_h^i(y)+p_h^{i-1}(y))}{2} \quad \text{and} \quad \delta^i = \frac{(y_h^i(u)+y_h^{i-1}(u))}{2} - \frac{(y_h^i+y_h^{i-1})}{2}.$$

We have that

$$\begin{aligned} J_2 = & \sum_{i=1}^{M} k_i\big(\gamma^i - I_h\gamma^i, B(u^{i-1/2}-u_h^{i-1/2})\big) + \sum_{i=1}^{M} k_i(\partial(y_h^i(u)-y_h^i), I_h\gamma^i) \\ & + \sum_{i=1}^{M} k_i\big[a_h(\delta^i, I_h\gamma^i) - a_h(\gamma^i, I_h\delta^i)\big] + \sum_{i=1}^{M} k_i a_h(\gamma^i, I_h\delta^i) \\ = & \sum_{i=1}^{M} k_i\big(\gamma^i - I_h\gamma^i, B(u^{i-1/2}-u_h^{i-1/2})\big) + \sum_{i=1}^{M} k_i\big[a_h(\delta^i, I_h\gamma^i) - a_h(\gamma^i, I_h\delta^i)\big] \\ & + \sum_{i=1}^{M} k_i(\partial(y_h^i(u)-y_h^i), I_h\gamma^i) + \sum_{i=1}^{M} k_i(\partial(p_h^i-p_h^i(y)), I_h\delta^i) \\ & + \sum_{i=1}^{M} k_i\Big(\frac{y_h^i(u)+y_h^{i-1}(u)}{2} - y^{i-1/2}, I_h\delta^i\Big) + \Big(-\sum_{i=1}^{M} k_i(\delta^i, I_h\delta^i)\Big) \\ \doteq & R_1+R_2+R_3+R_4+R_5+R_6. \end{aligned}$$

For the term R_1, using (4.1), Lemma 4.1 and Theorem 4.1, we can derive

$$\begin{aligned} R_1 = & \sum_{i=1}^{M} k_i\big(\gamma^i - I_h\gamma^i, B(u^{i-1/2}-u_h^{i-1/2})\big) \\ \le & C\sum_{i=1}^{M} k_i h \ \|\gamma^i\|_1 \ \|u^{i-1/2}-u_h^{i-1/2}\| \\ \le & C\sum_{i=1}^{M} k_i h \ \|p_h^i - p_h^i(y)\|_1 \ \|u^{i-1/2}-u_h^{i-1/2}\| \\ \le & Ch(h^2+k^2)|\|u-U_h\|| + Ch|\|u-U_h\||^2. \end{aligned}$$

In order to estimate the term R_2, by Lemma 4.1 and (4.3d), it follows that

$$\begin{aligned} R_2 &\le Ch\sum_{i=1}^{M} k_i\|\delta^i\|_1\ \|\gamma^i\|_1 \\ &\le Ch\sum_{i=1}^{M} k_i\|y_h^i(u)-y_h^i\|_1\ \|p_h^i-p_h^i(y)\|_1 \\ &\le Ch(|||u-U_h|||+C(h^2+k^2))|||u-U_h||| \\ &\le Ch(h^2+k^2)|||u-U_h|||+Ch|||u-U_h|||^2. \end{aligned}$$

For the term R_3 and R_4, Noticing $p_h^M-p_h^M(y)=0$ and $y_h^0(u)-y_h^0=0$, we can obtain

$$\begin{aligned} R_3+R_4 &= \sum_{i=1}^{M} k_i(\partial(y_h^i(u)-y_h^i),I_h\gamma^i)+\sum_{i=1}^{M} k_i(\partial(p_h^i-p_h^i(y)),I_h\delta^i) \\ &= \sum_{i=1}^{M}\big[(y_h^i(u)-y_h^i,I_h(p_h^i-p_h^i(y)))-(y_h^{i-1}(u)-y_h^{i-1},I_h(p_h^{i-1}-p_h^{i-1}(y)))\big] \\ &= 0. \end{aligned}$$

For the fifth term R_5, we have that

$$\begin{aligned} R_5 &= \sum_{i=1}^{M} k_i\Big(\frac{y_h^i(u)+y_h^{i-1}(u)}{2}-\frac{y^i+y^{i-1}}{2},I_h\delta^i\Big)+\sum_{i=1}^{M} k_i\Big(\frac{y^i+y^{i-1}}{2}-y^{1-1/2},I_h\delta^i\Big) \\ &\le C(h^2+k^2)|||u-U_h|||+Ck^2\|y_{tt}\|_{L^2(J,L^2)}|||u-U_h|||. \end{aligned}$$

Noticing $R_6\le 0$ and connecting T_1, T_2, J_1, J_2, R_1, R_2, R_3, R_4, R_5 and R_6, we can obtain (4.17) easily for sufficiently small h. □

Completing the proof of Theorem 4.2, we can obtain the following results.

Theorem 4.3. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of problems* (3.1a)-(3.1c) *and* (3.4a)-(3.4c), *respectively. Then there exists an* $h_0>0$ *such that for all* $0<h\le h_0$,

$$\|y^n-y_h^n\|+\|p^n-p_h^n\|\le C(h^2+k^2),\quad 0\le n\le M. \tag{4.18}$$

Proof. Using the triangle inequality, we have

$$\begin{aligned} \|y^n-y_h^n\| &\le \|y^n-y_h^n(u)\|+\|y_h^n(u)-y_h^n\|, \\ \|p^n-p_h^n\| &\le \|p^n-p_h^n(y)\|+\|p_h^n(y)-p_h^n\|. \end{aligned}$$

Lemma 4.1 implies that

$$\begin{aligned} \|y^n-y_h^n\| &\le \|y^n-y_h^n(u)\|+C|||u-U_h|||, \\ \|p^n-p_h^n\| &\le \|p^n-p_h^n(y)\|+C|||u-U_h|||+C(k^2+h^2). \end{aligned}$$

From Theorem 4.1 and Theorem 4.2, we can get (4.18) easily. □

Theorem 4.4. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of problems* (3.1a)-(3.1c) *and* (3.4a)-(3.4c)*, respectively. Then there exists an* $h_0>0$ *such that for all* $0<h\le h_0$

$$\|u^{n-1/2}-u_h^{n-1/2}\|\le C(h^2+k^2),\quad 0\le n\le M. \tag{4.19}$$

Proof. Introducing a projection (see, e.g., [12,14])

$$P_{[a,b]}(f(t,x))=\max\big(a,\min(b,f(t,x))\big), \tag{4.20}$$

we can denote the variational inequality (3.2c) by

$$u(t,x)=P_{[a,b]}\Big(-\frac{1}{\alpha}B^*p(t,x)\Big). \tag{4.21}$$

And the variational inequality (3.4c) is equivalent to

$$u_h^{n-1/2}=P_{[a,b]}\Big(-\frac{1}{\alpha}B^*\frac{p_h^n+p_h^{n-1}}{2}\Big).$$

It is obvious that

$$\begin{aligned}
|u^{n-1/2}-u_h^{n-1/2}|=&\Big|P_{[a,b]}\Big(-\frac{1}{\alpha}B^*p^{n-1/2}\Big)-P_{[a,b]}\Big(-\frac{1}{\alpha}B^*\frac{p_h^n+p_h^{n-1}}{2}\Big)\Big|\\
\le& C\Big|p^{n-1/2}-\frac{p_h^n+p_h^{n-1}}{2}\Big|\\
\le& C\Big(\Big|p^{n-1/2}-\frac{p^n+p^{n-1}}{2}\Big|+\Big|\frac{p^n+p^{n-1}}{2}-\frac{p_h^n+p_h^{n-1}}{2}\Big|\Big)\\
\le& Ck_n\int_{t_{n-1}}^{t_n}\|p_{tt}(s)\|ds+C\Big|\frac{p^n+p^{n-1}}{2}-\frac{p_h^n+p_h^{n-1}}{2}\Big|\\
\le& Ck^2\max_{s\in J}\Big\|p_{tt}(s)\Big\|+C\Big|\frac{p^n+p^{n-1}}{2}-\frac{p_h^n+p_h^{n-1}}{2}\Big|,
\end{aligned}$$

which implies (4.19) from Theorem 4.3. □

5 Numerical example

In order to test the theories of the previous sections, we present one numerical example to illustrate them. We use the scheme (3.4a)-(3.4c) and the algorithm presented in [15] to solve the problem.

Example 5.1. In this example, we investigate a distributed parabolic optimal control problem with Dirichlet boundary value condition.

$$\begin{aligned}
&\min_{u(t)\in K}\frac{1}{2}\int_0^1(\|y-y_d\|^2_{L^2(\Omega)}+\|u\|_{L^2(\Omega)})dt,\\
&y_t-\Delta y=f+u, && (x,t)\in\Omega\times J,\\
&y(x,t)=0, && (x,t)\in\partial\Omega\times J,\\
&y(x,0)=\sin(\pi x_1)\sin(\pi x_2), && x\in\Omega,
\end{aligned}$$

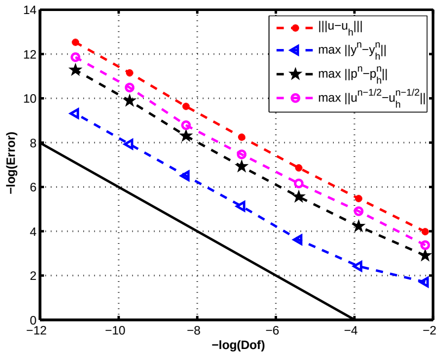

Figure 2: The convergent rates of the finite volume element approximations which are computed with $h=\sqrt{2}k$ (The slope of the solid line is -1).

where $\Omega=\{(x,y);\ 0\le x\le 1, 0\le y\le 1\}$, $J=(0,1]$, $f=-u=\max\{0.2,\min\{0.8,(1-t)\sin(\pi x_1)\sin(\pi x_2)\}\}$. The exact state

$$y(x,t)=e^{-2\pi^2 t}\sin(\pi x_1)\sin(\pi x_2).$$

As to the adjoint equation, $y_d=\sin(\pi x_1)\sin(\pi x_2)(e^{-2\pi^2 t}+1-2\pi^2(t-1))$. The exact adjoint state $p(x,t)=(t-1)\sin(\pi x_1)\sin(\pi x_2)$.

In our numerical experiments, for an integer N, we use uniform spatial and time partition with the step size $h=\sqrt{2}/N$ and $k=1/N$ to check the corresponding convergent rates.

The errors for the numerical solutions are reported in Table 1, and the corresponding convergent rates are showed in Fig. 2. For the last column of Table 1, it has the relation that is showed in Table 2.

In Table 1, the ith line is almost four times of the $(i+1)$th line, $i=1,2,\cdots,6$, which means that the convergent rates are $\mathcal{O}(h^2+k^2)$. In Fig. 2, the slope of the solid line is -1, which means the convergent rates are also $\mathcal{O}(h^2+k^2)$. All in all, from Table 1 and Fig. 2, the convergent orders match the theories derived in the previous sections.

Table 1: Numerical results with $h=\sqrt{2}k$.

$\lvert\lvert\lvert u-u_h\rvert\rvert\rvert$	$\max_{0\le n\le M}\lVert y^n-y_h^n\rVert$	$\max_{0\le n\le M}\lVert p^n-p_h^n\rVert$	$\max\lVert u^{n-1/2}-u_h^{n-1/2}\rVert$	Dof
0.01865771834056	0.18170420067366	0.05510738615902	0.03430988025015	9
0.00417734045923	0.08916381929704	0.01466851715310	0.00741554151099	49
0.00104721146220	0.02686325341669	0.00387209697601	0.00212123649630	225
0.00026168616068	0.00593982368997	0.00098683344957	0.00056888113768	961
0.00006539284611	0.00149899432175	0.00024801812421	0.00015234441836	3969
0.00001437882496	0.00036019944944	0.00005090382369	0.00002796642899	16641
0.00000360699850	0.00008992267101	0.00001267050545	0.00000706659612	66049

Table 2: The relation between Dof and h.

Dof	9	49	225	961	3969	16641	66049
$h/\sqrt{2}$	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$	$1/128$	$1/256$

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