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Analyses of the Dispersion Overshoot and Inverse Dissipation of the High-Order Finite Difference Scheme

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Abstract. Analyses were performed on the dispersion overshoot and inverse dissipation of the high-order finite difference scheme using Fourier and precision analysis. Schemes under discussion included the pointwise- and staggered-grid type, and were presented in weighted form using candidate schemes with third-order accuracy and three-point stencil. All of these were commonly used in the construction of difference schemes. Criteria for the dispersion overshoot were presented and their critical states were discussed. Two kinds of instabilities were studied due to inverse dissipation, especially those that occur at lower wave numbers. Criteria for the occurrence were presented and the relationship of the two instabilities was discussed. Comparisons were made between the analytical results and the dispersion/dissipation relations by Fourier transformation of typical schemes. As an example, an application of the criteria was given for the remedy of inverse dissipation in Weirs & Martín's third-order scheme.

AMS subject classifications: 76M20, 74P99

Key words: High-order difference scheme, dispersion overshoot, inverse dissipation.

1 Introduction

High-order schemes are widely used in direct numerical simulation and large eddy simulation to resolve turbulent structures with broad length scales. Numerical simulation of shock/turbulent boundary layer interaction is an example that requires high order accuracy for capturing the broad length scales and the capability of shock capturing. The main advantage of the high-order scheme is the lower truncation error, which is obtained

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through precision analyses. As shown in [1], the spectral characteristic of the scheme is another important attribute. It is not unusual that relative low-order schemes can have better dispersion/dissipation relation than that of higher-order ones.

Bandwidth optimization refers to the technique for improving the dispersion and/or dissipation features of difference schemes, which are derived through Fourier analysis. The methodology is to sacrifice the order of the scheme and use the available free parameters for optimization. Lele [2] and Tam [3] pioneered studies in this aspect from different considerations. Several variants of Tam's methods were proposed subsequently [4–6].

The optimization of the dispersion relation is to make the real part of the modified wave number ($\Re(\kappa')$) to distribute as close as possible along the theoretical one (κ). Generally, the dispersion curve of the standard center or upwinding-biased pointwise scheme moves below the theoretical straight line and drops to zero at π. Optimization tries to push the deviation of the curve to the line toward the higher band of the scaled wave number. Lockard [4] stressed that the moving of the curve above the theoretical one might happened in this process ("a noticeable overshoot"), which is called as dispersion overshoot in the paper. Because dispersion overshoot implies the possibility of excessive "deviation from the correct result for smaller values" (i.e., scaled wave numbers) like that in "Tam's operators" [4], which might result in phase errors for wave propagation, it is natural that attempts be made to solve the problem. A modified objective function was given by Lockard [4] to try to alleviate the phenomenon, but no analysis was given regarding the cause of the overshoot.

The numerical dissipation is another issue concerned by the optimization of upwinding-biased schemes. Usually some dissipation is necessary for numerical stability, especially in the case of shock capturing. As proposed by Adams & Shariff [5], the stability criterion is that, the imaginary part of the modified wave number satisfies: $\Im(\kappa') \leq 0$. This criterion has often been used by simply checking or requiring that the dissipation at π (called as $\pi_{dissipation}$ in the paper) be less than zero, and assuming that the criterion is satisfied in the region $[0,\pi]$ also. No further attempt is made to thoroughly check if the criterion is satisfied throughout the region.

To test Lockard's method, we used it to perform optimizations and obtained a third-order scheme (referred as LKD3), which used the same candidate schemes as that of the fifth-order WENO scheme but had different weights. The result showed that the optimized scheme still had minor dispersion overshoot, which indicated that the optimization method proposed by Lockard [4] might not truly solve the problem. In the meanwhile, when we checked the dissipation curve of Weirs & Martín's third-order scheme [6,7] (referred as WM3), we found that the inverse dissipation existed at lower wave numbers. To explore above issues, we analyzed the dispersion overshoot and the inverse dissipation, and found the cause and criteria for the occurrence. The purpose of this study is not to develop a specific scheme, but to provide useful theoretical references to support bandwidth optimization of high-order schemes.

The paper is organized as follows. In Section 2, we report our investigations in pointwise high-order finite difference schemes, and present the analytical results; in Section 3,