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A Meshless Regularization Method for a Two-Dimensional Two-Phase Linear Inverse Stefan Problem

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Abstract. In this paper, a meshless regularization method of fundamental solutions is proposed for a two-dimensional, two-phase linear inverse Stefan problem. The numerical implementation and analysis are challenging since one needs to handle composite materials in higher dimensions. Furthermore, the inverse Stefan problem is ill-posed since small errors in the input data cause large errors in the desired output solution. Therefore, regularization is necessary in order to obtain a stable solution. Numerical results for several benchmark test examples are presented and discussed.

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Key words: Heat conduction, method of fundamental solutions (MFS), inverse Stefan problem, two-phase change.

1 Introduction

Heat conduction Stefan problems with phase change in multiple dimensions are of importance in several industrial applications in continuous casting of steel, welding processes, crystal and biofilm growth, etc. The classical direct Stefan problem which requires determining both the temperature and the free boundary can become tedious and complicated in the case of multi-dimensional multi-phase models. This fact has motivated researchers to consider inverse Stefan problems in which the free boundary is known and the boundary and/or initial data are unknown [4, 6]. This inverse problem which has application in the technology of refining a material by means of recrystalisation [16],

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is difficult to solve since, as a non-characteristic Cauchy problem, it is ill-posed [2, 4, 9]. Although there exists an extensive literature on one-phase one- and two-dimensional inverse Stefan problems, the two-dimensional two-phase case has been considerably less examined. Prior to this study, [1] regularized such an inverse and ill-posed problem by means of a convolution equation, but the domain considered in their paper is infinite. In this paper, we develop a meshless regularized numerical method of fundamental solutions (MFS) for solving a two-dimensional two-phase linear inverse Stefan problem. In doing so, we extend the recent meshless method of fundamental solutions proposed in [11, 13] for the one-dimensional two-phase and two-dimensional one-phase inverse linear Stefan problems, respectively, to the two-dimensional two-phase change case. Further applications of the MFS to inverse problems can be found in the survey paper [14].

2 Mathematical formulation

In this section, we extend some of the notation and mathematical setup of [5] from the one-phase to the two-phase situation. Let $l>0$, $T>0$ and for $t\in[0,T]$ define the liquid (water) zone

$$\Omega_1(t)=\{(x,y)\in\mathbb{R}^2 \mid 0<x<s(y,t),\ 0<y<1\},$$

and the solid (ice) zone

$$\Omega_2(t)=\{(x,y)\in\mathbb{R}^2 \mid s(y,t)<x<l,\ 0<y<1\},$$

where the liquid-solid interface $s(y,t)\in(0,l)$ is known and given. The boundaries $\partial\Omega_i(t)=\Gamma_i(t)\cup\Sigma(t)$, where

$$\Sigma(t)=\{(x,y)\in\mathbb{R}^2 \mid x=s(y,t),\ 0<y<1\}$$

and $\Gamma_i(t)=\partial\Omega_i(t)\backslash\Sigma(t)$, for $i=1,2$. Denote also $\Omega(t)=\Omega_1(t)\cup\Sigma(t)\cup\Omega_2(t)$, so that $\partial\Omega(t)=\Gamma_1(t)\cup\Gamma_2(t)$. The whole solution domain of each piece of the composite bi-material, for $i=1,2$, are denoted by $\Omega_i=\bigcup_{t\in(0,T]}\Omega_i(t)$, and we observe that the boundary $\partial\Omega_i$ of Ω_i consists of the "bottom"

$$\overline{\Omega_1(0)}=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le s(y,0),\ 0\le y\le 1\},$$
$$\overline{\Omega_2(0)}=\{(x,y)\in\mathbb{R}^2 \mid s(y,0)\le x\le l,\ 0\le y\le 1\},$$

the "top"

$$\overline{\Omega_1(T)}=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le s(y,T),\ 0\le y\le 1\},$$
$$\overline{\Omega_2(T)}=\{(x,y)\in\mathbb{R}^2 \mid s(y,T)\le x\le l,\ 0\le y\le 1\},$$

the interface boundary $\Sigma=\bigcup_{t\in(0,T)}\Sigma(t)$, and the "fixed" boundary $\Gamma_i=\bigcup_{t\in(0,T)}\Gamma_i(t)$. We assume that the interface $s\in(0,l)$ is a known and sufficiently smooth function.