Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **5**, No. 6, pp. 872-884

DOI: 10.4208/aamm.12-m12118
December 2013

A Collocation Method for Solving Fractional Riccati Differential Equation

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Received 16 September 2012; Accepted (in revised version) 10 May 2013

Available online 6 September 2013

Abstract. In this article, we have introduced a Taylor collocation method, which is based on collocation method for solving fractional Riccati differential equation. The fractional derivatives are described in the Caputo sense. This method is based on first taking the truncated Taylor expansions of the solution function in the fractional Riccati differential equation and then substituting their matrix forms into the equation. Using collocation points, the systems of nonlinear algebraic equation is derived. We further solve the system of nonlinear algebraic equation using Maple 13 and thus obtain the coefficients of the generalized Taylor expansion. Illustrative examples are presented to demonstrate the effectiveness of the proposed method.

AMS subject classifications: 34A08, 656N35, 30K05

Key words: Riccati equation, fractional derivative, collocation method, generalized Taylor series, approximate solution.

1 Introduction

The concept of fractional or non-integer order derivation and integration can be traced back to the genesis of integer order calculus itself [1, 2]. Fractional calculus has become the focus of interest for many researchers in different disciplines of science and technology. The fractional differential equations (FDEs) have received considerable interest in recent years. FDEs have shown to be adequate models for various physical phenomena in areas like damping laws, diffusion processes, etc. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [3], the fluid-dynamic traffic model with fractional derivatives [4], psychology [5] and etc. [6–9].

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In this paper, we present numerical and analytical solutions for the fractional Riccati differential equation

$$D^{\alpha}y(x)=A(x)+B(x)y(x)+C(x)y^2(x), \quad x>0, \quad 0<\alpha\leq 1, \tag{1.1}$$

subject to the initial conditions

$$y(0)=\lambda, \tag{1.2}$$

where $A(x)$, $B(x)$ and $C(x)$ are given functions, α is a parameter describing the order of the fractional derivative. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha=1$ the fractional equation reduces to the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems [10]. The existing literature on fractional differential equations tends to focus on particular values for the order α. The value $\alpha=0.5$ is especially popular. This is because in classical fractional calculus, many of the model equations developed used these particular orders of derivatives [11]. In modern applications (see, e.g., [12]) much more general values of the order an appear in the equations and therefore one needs to consider numerical and analytical methods to solve differential equations of arbitrary order. This equation is solve numerically in [13–15]. In [13], it is given numerical solution of approximate solution of linear fractional differential equations with variable coefficients by collocation method. In [14], a modification of He's homotopy perturbation method is presented. In this method, which does not require a small parameter in an equation, a homotopy with an imbedding parameter $p\in[0,1]$ is constructed. In [15], it is implemented a relatively new analytical technique, the Adomian decomposition method. The solution takes the form of a convergent series with easily computable components. The diagonal Pade approximants are effectively used in the analysis to capture the essential behavior of the solution.

We seek by collocation method the approximate solution of Eq. (1.1) under the condition Eq. (1.2) using the fractional Taylor series

$$y_N(x)=\sum_{i=0}^{N}\frac{(x-c)^{i\alpha}}{\Gamma(i\alpha+1)}(D^{i\alpha}y(x))_{x=c}, \tag{1.3}$$

where $0<\alpha\leq 1$ and $D^{i\alpha}y(x)\in C(a,b]$. This method transforms each part of equation into a matrix form, and using the collocation points

$$x_i=\frac{i}{N}, \quad i=0,1,\cdots,N, \tag{1.4}$$

we derive the nonlinear algebraic equation. Solving this equation, we obtained the coefficients of the generalized Taylor series and thus the approximate solutions for various N. Recently, collocation method has become a very useful technique for solving equations. For instance, some authors gave the numerical studies for solving linear differential difference equations [16], Volterra integral equations [17], linear integro-differential equations [18–21], Abel equation [22], nonlinear differential equations [23] by some special functions. Chen et al. [24] presented the Kansa method, which belonged to the RBF collocation method, for solving fractional diffusion equations. Fu et al. [25] have given a novel boundary-type RBF collocation approach, Laplace transformed boundary particle method, for solving time fractional diffusion equations. Brunner et al. [26] presented a RBF collocation method, which includes geometric time grid relaxation and adaptive kernel selection, for solving 2D fractional subdiffusion problems.

2 Basic definitions

In this section, we give the generalized Taylor formula and the definitions related [27]. There are several definitions of a fractional derivative of order $\alpha > 0$ [1, 2]. The two most commonly used definitions are the Riemann-Liouville and Caputo. Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. Riemann-Liouville fractional derivative is computed in the reverse order. Therefore, Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem, but Riemann-Liouville fractional derivative allows initial conditions in terms of fractional integrals and their derivatives.

Definition 2.1. A real function $f(x)$, $x>0$, is said to be in space C_μ, $\mu\in\mathbb{R}$ if there exist a real number p $(p>\mu)$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)}\in C_\mu$, $m\in\mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional derivative of order α with respect to the variable x and with the starting point at $x=a$ is

$$ {}_aD_x^\alpha f(x)=\begin{cases} \dfrac{1}{\Gamma(-\alpha+m+1)}\dfrac{d^{m+1}}{dx^{m+1}}\displaystyle\int_a^x (x-\tau)^{m-\alpha}f(\tau)d\tau, & 0\le m\le\alpha<m+1, \\ \dfrac{d^m}{dx^m}, & \alpha=m+1\in\mathbb{N}. \end{cases} $$

Definition 2.3. The Riemann-Liouville fractional integral of order α is

$$ {}_aD_x^{-\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}f(\tau)d\tau, \quad p>0. $$

Definition 2.4. The Caputo fractional derivative of $f(x)$ is defined as

$$ D^\alpha f(x)=\frac{1}{\Gamma(n\alpha)}\int_0^x (x-\tau)^{n-\alpha-1}f^n(\tau)d\tau, $$

for $n-1<\alpha\le n$, $n\in\mathbb{N}$, $x>0$, $f\in C_{-1}^n$.

For the Caputo derivative we further have: $D^{\alpha}C=0$, as C is a constant,

$$D^{\alpha}x^{n}=\begin{cases}0, & n\in\mathbb{N}, \quad n<\lceil\alpha\rceil,\\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}, & n\in\mathbb{N}, \quad n<\lfloor\alpha\rfloor.\end{cases}$$

Theorem 2.1. *Supposing $D^{k\alpha}f(x)\in C(0,b]$, for $i=0,1,\cdots,n+1$, where $0<\alpha\le 1$, then we have* [27]

$$f(x)=\sum_{i=0}^{n}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}(D^{i\alpha}f(x))_{x=a}+\frac{(D^{n+1}f(x))(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha},$$

where $a\le\xi\le x$, for $\forall x\in(a,b]$ and

$$D^{n\alpha}=D^{\alpha}D^{\alpha}D^{\alpha}\cdots D^{\alpha}(n\ times).$$

3 Fundamental relations

In this section, we consider the fractional Riccati differential equations Eq. (1.1). We use the Taylor matrix method [16–23] to find the truncated Taylor series expansions of each term in expression at $x=c$ and their matrix representations for solving $\alpha-th$ order non-linear part. We first consider the solution $y_N(x)$ of Eq. (1.1) defined by a truncated Taylor series Eq. (1.3). Then, we have the matrix form of the approximate solution $y_N(x)$

$$[y_N(x)]=\mathbf{T}(x)\mathbf{A}=\mathbf{X}(x)\mathbf{M}_0\mathbf{A}, \tag{3.1}$$

where

$$\mathbf{X}(x)=\begin{bmatrix}1 & (x-c)^{\alpha} & (x-c)^{2\alpha} & \cdots & (x-c)^{N\alpha}\end{bmatrix},$$

$$\mathbf{M}_0=\begin{bmatrix}\frac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{\Gamma(\alpha+1)} & 0 & \cdots & 0\\ 0 & 0 & \frac{1}{\Gamma(2\alpha+1)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{\Gamma(N\alpha+1)}\end{bmatrix}, \quad \mathbf{A}=\begin{bmatrix}D^{0\alpha}y(c)\\ D^{1\alpha}y(c)\\ D^{2\alpha}y(c)\\ \vdots\\ D^{N\alpha}y(c)\end{bmatrix}.$$

The matrix representation of $D^{\alpha}y_N(x)$ becomes

$$D^{\alpha}y_N(x)=D^{\alpha}\mathbf{X}(x)\mathbf{M}_0\mathbf{A},$$

where

$$\begin{aligned} D^{\alpha}\mathbf{X}(x) &= \begin{bmatrix} D^{\alpha}1 & D^{\alpha}(x-c)^{\alpha} & D^{\alpha}(x-c)^{2\alpha} & \cdots & D^{\alpha}(x-c)^{N\alpha} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \dfrac{\Gamma(\alpha+1)}{\Gamma(1)} & \dfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(x-c)^{\alpha} & \cdots & \dfrac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)}(x-c)^{(N-1)\alpha} \end{bmatrix} \\ &= \mathbf{X}(x)\mathbf{M}_1, \end{aligned}$$

where

$$\mathbf{M}_1 = \begin{bmatrix} 0 & \dfrac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus the matrix representation of fractional differential term is

$$D^{\alpha}y_N(x) = \mathbf{X}(x)\mathbf{M}_1\mathbf{M}_0\mathbf{A}. \tag{3.2}$$

Moreover, from [22,23]

$$\mathbf{Y}^m = \mathbf{Y}^{m-1}\overline{\mathbf{Y}}, \tag{3.3}$$

where

$$\mathbf{Y}^{m-1}(x) = \begin{bmatrix} y^{m-1}(x) \\ y^{m-1}(x) \\ \vdots \\ y^{m-1}(x) \end{bmatrix}, \quad \overline{\mathbf{Y}}(x) = \begin{bmatrix} y(x) & 0 & \cdots & 0 \\ 0 & y(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y(x) \end{bmatrix},$$

and using collocation points in Eq. (3.1) leads to

$$\mathbf{Y}^m = \overline{\mathbf{T}\mathbf{A}}, \tag{3.4}$$

where

$$\overline{\mathbf{T}}(x_i) = \begin{bmatrix} \mathbf{T}(x_i) & 0 & \cdots & 0 \\ 0 & \mathbf{T}(x_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T}(x_i) \end{bmatrix}, \quad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 \\ 0 & \mathbf{A} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A} \end{bmatrix}.$$

Reference to Eqs. (3.3) and (3.4) gives the following relation

$$y^2(x_i) = y(x_i)y(x_i) = (\overline{\mathbf{T}\mathbf{A}})\mathbf{X}(x_i)\mathbf{M}_0\mathbf{A}. \tag{3.5}$$

Hence, the fundamental matrix relation of the Eq. (1.1) is given by

$$\Big(\mathbf{X}(x)\mathbf{M}_1\mathbf{M}_0 - B(x)\mathbf{X}(x)\mathbf{M}_0 - C(x)(\overline{\mathbf{T}\mathbf{A}})\mathbf{X}(x)\mathbf{M}_0\Big)\mathbf{A} = A(x). \tag{3.6}$$

Similarly, we obtain matrix representation of the condition in Eq. (1.2)

$$U_(0) = \mathbf{X}(0)\mathbf{M}_0\mathbf{A} = \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_N \end{bmatrix} = \begin{bmatrix} \lambda \end{bmatrix}. \tag{3.7}$$

4 Method of solution

Using collocation points in Eq. (1.4), we can write the Eq. (3.6) as

$$\Big(\mathbf{X}(x_i)\mathbf{M}_1\mathbf{M}_0 - B(x_i)\mathbf{X}(x_i)\mathbf{M}_0 - C(x_i)(\overline{\mathbf{T}\mathbf{A}})\mathbf{X}(x_i)\mathbf{M}_0\Big)\mathbf{A} = A(x_i) \tag{4.1}$$

or in a matrix-vector form

$$\Big(\mathbf{X}\mathbf{M}_1\mathbf{M}_0 - \mathbf{B}\mathbf{X}\mathbf{M}_0 - \mathbf{C}(\overline{\overline{\mathbf{T}}}\,\overline{\mathbf{A}})\mathbf{X}\mathbf{M}_0\Big)\mathbf{A} = \mathbf{F}, \tag{4.2a}$$

$$\mathbf{X} = \begin{bmatrix} 1 & (x_0-c)^{\alpha} & (x_0-c)^{2\alpha} & \cdots & (x_0-c)^{N\alpha} \\ 1 & (x_1-c)^{\alpha} & (x_1-c)^{2\alpha} & \cdots & (x_1-c)^{N\alpha} \\ 1 & (x_2-c)^{\alpha} & (x_2-c)^{2\alpha} & \cdots & (x_2-c)^{N\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (x_N-c)^{\alpha} & (x_N-c)^{2\alpha} & \cdots & (x_N-c)^{N\alpha} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} A(x_0) \\ A(x_1) \\ A(x_2) \\ \vdots \\ A(x_N) \end{bmatrix}, \tag{4.2b}$$

$$\overline{\overline{\mathbf{T}}} = \begin{bmatrix} \overline{\mathbf{T}}(x_0) & 0 & 0 & \cdots & 0 \\ 0 & \overline{\mathbf{T}}(x_1) & 0 & \cdots & 0 \\ 0 & 0 & \overline{\mathbf{T}}(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \overline{\mathbf{T}}(x_N) \end{bmatrix}, \tag{4.2c}$$

$$\mathbf{B} = \begin{bmatrix} B(x_0) & 0 & 0 & \cdots & 0 \\ 0 & B(x_1) & 0 & \cdots & 0 \\ 0 & 0 & B(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B(x_N) \end{bmatrix}, \tag{4.2d}$$

$$\mathbf{C} = \begin{bmatrix} C(x_0) & 0 & 0 & \cdots & 0 \\ 0 & C(x_1) & 0 & \cdots & 0 \\ 0 & 0 & C(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C(x_N) \end{bmatrix}. \tag{4.2e}$$

Hence, the fundamental matrix equation (4.2a) corresponding to Eq. (1.1) can be written in

$$\mathbf{W}\mathbf{A} = \mathbf{F} \;\text{ or }\; [\mathbf{W};\mathbf{F}], \quad \mathbf{W} = [w_{ij}], \quad i,j = 0,1,\cdots,N, \tag{4.3}$$

where

$$\mathbf{W}=\mathbf{X}\mathbf{M}_1\mathbf{M}_0-\mathbf{B}\mathbf{X}\mathbf{M}_0-\mathbf{C}(\overline{\overline{\mathbf{T}\mathbf{A}}})\mathbf{X}\mathbf{M}_0.$$

To obtain the solution of Eq. (1.1) with condition Eq. (1.2), by substituting the row vector Eq. (3.7) in the last row of the matrix in Eq. (4.2a), we obtain the new augmented matrix;

$$[\mathbf{W}^*;\mathbf{F}^*]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & A(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & A(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{(N-2)0} & w_{(N-2)1} & \cdots & w_{(N-2)N} & ; & A(x_{N-2}) \\ w_{(N-1)0} & w_{(N-1)1} & \cdots & w_{(N-1)N} & ; & A(x_{N-1}) \\ u_0 & u_1 & \cdots & u_N & ; & \lambda \end{bmatrix}$$

or the corresponding matrix equation

$$\mathbf{W}^*\mathbf{A}=\mathbf{F}^*. \tag{4.4}$$

In doing so, we obtain a system of $(N+1)$ nonlinear algebraic equations with $(N+1)$ unknown generalized Taylor coefficients. If $det(\mathbf{W}^*)\neq 0$, we can write Eq. (4.4) as

$$\mathbf{A}=(\mathbf{W}^*)^{-1}\mathbf{F}^*$$

and the matrix $\mathbf{A}$ is uniquely determined. Therefore, the approximate solution is given by the truncated fractional Taylor series

$$y_N(x)=\sum_{i=0}^{N}\frac{(x-c)^{i\alpha}}{\Gamma(i\alpha+1)}(D^{i\alpha}y(x))_{x=c}.$$

We can easily check the accuracy of the method. Since the truncated fractional Taylor series Eq. (1.3) is an approximate solution of Eq. (1.1), when the solution $y_N(x)$ and its derivatives are substituted in Eq. (1.1), the resulting equation must be satisfied approximately; that is, for $x=x_q\in[0,1]$, $q=0,1,2,\cdots$,

$$E_N(x_q)=|D^{\alpha}y(x_q)-A(x_q)-B(x_q)y(x_q)-C(x_q)y^2(x_q)|\cong 0.$$

5 Examples

In order to illustrate the effectiveness of the method proposed in this paper, several numerical examples are carried out in this section. In the followings, absolute errors between N-th order approximate values y_N and the corresponding exact values y_{ex} $N_e=|y_N-y_{ex}|$ are determined and all computations are performed with the computer algebraic system in Maple 13.

Example 5.1. Consider the following fractional Riccati equation:

$$D^{\alpha}y(x)=y^2(x)-x^2y(x)+\frac{\Gamma(3)}{\Gamma(2.5)}x^{1.5},\quad x>0,$$

with initial conditions

$$y(0)=0.$$

Then, $A(x)=\frac{\Gamma(3)}{\Gamma(2.5)}x^{1.5}$, $B(x)=-x^2$, $C(x)=1$. We assume that $\alpha=1/2$ and use the Taylor series, for $c=0$, $N=4$

$$y_4(x)=\sum_{i=0}^{4}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}(D^{i\alpha}y(x))_{x=0}$$

as well as the collocation points

$$x_0=0,\quad x_1=\frac{1}{4},\quad x_2=\frac{2}{4},\quad x_3=\frac{3}{4},\quad x_4=1.$$

Fundamental matrix relation of this problem is

$$\Big(\mathbf{X}\mathbf{M}_1\mathbf{M}_0-\mathbf{B}\mathbf{X}\mathbf{M}_0-\mathbf{C}(\overline{\overline{\mathbf{T}}}\,\overline{\mathbf{A}})\mathbf{X}\mathbf{M}_0\Big)\mathbf{A}=\mathbf{F}, \tag{5.1}$$

where

$$\mathbf{X}=\begin{bmatrix}1&0&0&0&0\\1&\frac{1}{\sqrt{4}}&\frac{1}{4}&\frac{\sqrt{4}}{16}&\frac{1}{16}\\1&\frac{1}{\sqrt{2}}&\frac{1}{2}&\frac{\sqrt{2}}{4}&\frac{1}{4}\\1&\frac{\sqrt{3}}{\sqrt{4}}&\frac{3}{4}&\frac{3\sqrt{3}}{4\sqrt{4}}&\frac{9}{16}\\1&1&1&1&1\end{bmatrix},\qquad \mathbf{B}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix}0&0&0&0&0\\0&0.0625&0&0&0\\0&0&0.25&0&0\\0&0&0&0.5625&0\\0&0&0&0&1\end{bmatrix},\qquad \mathbf{M}_0=\begin{bmatrix}1&0&0&0&0\\0&\frac{2}{\sqrt{\pi}}&0&0&0\\0&0&1&0&0\\0&0&0&\frac{4}{3\sqrt{\pi}}&0\\0&0&0&0&\frac{1}{2}\end{bmatrix},$$

$$\mathbf{M}_1=\begin{bmatrix}0&1&0&0&0\\0&0&\frac{2}{\sqrt{\pi}}&0&0\\0&0&0&1&0\\0&0&0&0&\frac{4}{3\sqrt{\pi}}\\0&0&0&0&0\end{bmatrix},\qquad \mathbf{F}=\begin{bmatrix}0.000000\\0.188063\\0.531923\\0.977205\\1.504505\end{bmatrix}.$$

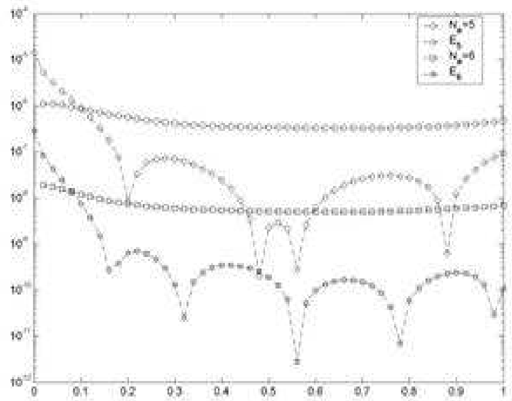

Figure 1: Comparison of the absolute errors and error estimation functions for Example 5.1.

Also, we have the matrix representation of conditions

$$y(0)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{A}=\begin{bmatrix} 0 \end{bmatrix}$$

and so we solve the Eq. (5.1) and obtain the coefficients of the Taylor series

$$\mathbf{A}=\begin{bmatrix} 0 & -0.478771e-4 & 0.189414e-3 & -0.307615e-3 & 2.000189 \end{bmatrix}.$$

Hence, for $N=4$, the approximate solution of Example 5.1 is given by

$$y_4(x)=0.957543e-3\frac{x^{\frac{1}{2}}}{\sqrt{\pi}}+0.189414e-3x-0.410153e-4\frac{x^{\frac{3}{2}}}{\sqrt{\pi}}+1.000094x^2.$$

Comparison of numerical results with the exact solution are shown in Table 1 and plotted in Fig. 1 for various N.

Table 1: Numerical results for Example 5.1.

x	Exact solution	$N=4$	$N_e=4$	$N=5$	$N_e=5$	$N=6$	$N_e=6$
0.0	0.000	0.000000	0.000e-0	0.000000	0.000e-0	0.000000	0.000e-0
0.2	0.040	0.039985	0.150e-4	0.040000	0.553e-6	0.040000	0.740e-8
0.4	0.160	0.159993	0.700e-5	0.160000	0.352e-6	0.160000	0.530e-8
0.6	0.360	0.359996	0.446e-5	0.360000	0.330e-6	0.360000	0.497e-8
0.8	0.640	0.639996	0.446e-5	0.640000	0.337e-6	0.640000	0.528e-8
1.0	1.000	0.999998	0.219e-5	1.000000	0.464e-6	1.000000	0.673e-8

Example 5.2. Let us consider the following fractional Riccati equation [14]

$$D^{\alpha}y(x)=y^2(x)+1$$

subject to the initial condition

$$y(0)=1.$$

Table 2: Numerical results for Example 5.2.

		PM				HPM	
x	Exact solution	$N=6$	$N_e=6$	$N=12$	$N_e=12$	Numerical solution	Absolute errors
0.0	0.000000	0.000000	0.000e-0	0.000000	0.000e-0	0.000000	0.000e-0
0.1	0.099669	0.099667	0.135e-5	0.099667	0.467e-9	0.099668	0.100e-5
0.2	0.197375	0.197377	0.169e-5	0.197375	0.411e-9	0.197375	0.000e-0
0.3	0.291313	0.291313	0.878e-6	0.291313	0.395e-9	0.291312	0.100e-5
0.4	0.379949	0.379949	0.103e-5	0.379949	0.371e-9	0.379944	0.500e-5
0.5	0.462117	0.462118	0.151e-5	0.462117	0.338e-9	0.462078	0.390e-4
0.6	0.537050	0.537050	0.511e-6	0.537050	0.308e-9	0.536857	0.193e-3
0.7	0.604378	0.604367	0.140e-5	0.604378	0.274e-9	0.603631	0.737e-3
0.8	0.664037	0.664041	0.425e-5	0.664041	0.229e-9	0.661706	0.133e-3
0.9	0.716298	0.716292	0.163e-5	0.716298	0.394e-9	0.709919	0.637e-2
1.0	0.761594	0.761466	0.127e-3	0.761594	0.121e-7	0.746032	0.155e-1

Then, $A(x)=1$, $B(x)=0$, $C(x)=1$. Fundamental matrix relation of this problem is

$$\Big(\mathbf{X}\mathbf{M}_1\mathbf{M}_0-\mathbf{C}(\overline{\overline{\mathbf{T}}}\,\overline{\mathbf{A}})\mathbf{X}\mathbf{M}_0\Big)\mathbf{A}=\mathbf{F}.$$

Also, we have the matrix representation of conditions;

$$y(0)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{A}=\begin{bmatrix} 0 \end{bmatrix}.$$

The exact solution, when $\alpha=1$, is

$$y(x)=\frac{e^{2x}-1}{e^{2x}+1}.$$

We approximately solve the fractional Riccati equation for $N=12$ and obtained the approximate solution for $\alpha=1$;

$$\begin{aligned} y_{12}(x)=&x-0.559171e-6x^2-0.333319x^3-0.155476e-3x^4+0.134388x^5-0.467422e-2x^6\\ &-0.398472e-1x^7-0.295050e-1x^8+0.640148e-1x^9-0.390402x^{10}+0.109027x^{11}. \end{aligned}$$

Table 2 shows the approximate solutions for Example 5.2 obtained by the Present Method (PM) compared to those from Homotopy Perturbation Method (HPM) [14] for $\alpha=1$. We give the comparison of the absolute errors for PM and HPM in Fig. 2. From the numerical results in Table 2 and Fig. 2, PM is able to achieve higher accuracy than HPM. Additionally, Fig. 3 displays the comparison of the absolute errors and error estimation functions for $N=6,12$.

6 Conclusions

In this study, we present a Taylor collocation method for the numerical solutions of fractional Riccati equation. This method transforms the fractional Riccati differential equation into a set of equations. The desired approximate solutions can be determined by

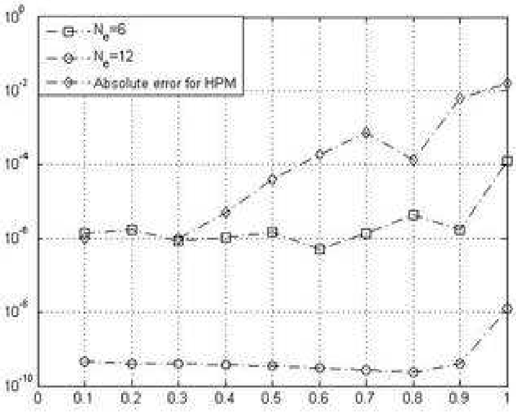

Figure 2: Comparison of the absolute errors for PM and HPM.

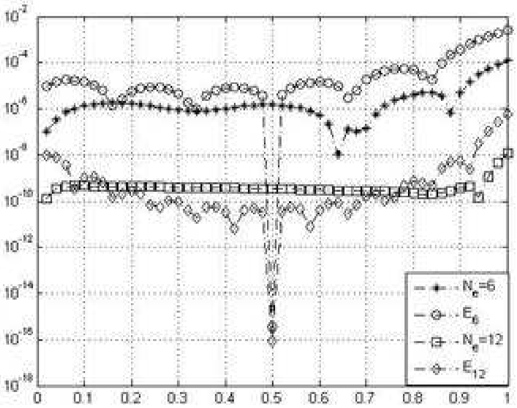

Figure 3: Comparison of the absolute errors and error estimation functions for PM.

solving the resulting system, which can be effectively computed using symbolic computing codes in Maple 13. Numerical results show that the Taylor collocation method can be successfully applied to solve fractional Riccati differential equation at high accuracy.

Acknowledgments

We are very much indebted unknown reviewer for carefully reading the paper, comments and remarks which led to improved presentation and quality of this paper.

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