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The Eigenfunctions and Exact Solutions of Discrete mKdV Hierarchy with Self-Consistent Sources via the Inverse Scattering Transform

Q. Li[1,*], J. B. Zhang[2] and D. Y. Chen[3]

[1] *State Key Laboratory Breeding Base of Nuclear Resources and Environment, Department of Mathematics, East China Institute of Technology, Nanchang 330013, China*
[2] *School of Mathematics and Statistics, Jiangsu Normal University, Jiangsu 221116, China*
[3] *Department of Mathematics, Shanghai University, Shanghai 200444, China*

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Abstract. Another form of the discrete mKdV hierarchy with self-consistent sources is given in the paper. The self-consistent sources is presented only by the eigenfunctions corresponding to the reduction of the Ablowitz-Ladik spectral problem. The exact soliton solutions are also derived by the inverse scattering transform.

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1 Introduction

Soliton equations with self-consistent sources model some phenomena in hydrodynamics, plasma physics, solid state physics [1–9]. For reason of an arbitrary time-dependence in dispersion relation, the self-consistent sources result in some interesting dynamical characters [10, 11]. In terms of solutions, some methods have been used to investigate some soliton equations including those soliton equations with self-consistent sources, such as the inverse scattering transform (IST), Bäcklund transformation, Darboux transformation, bilinear method [12–23]. Recently, the transformed rational function method, the multiple exp-function algorithm, the linear superposition principle and the invariant subspace method are also developed to solve some soliton equations [24–27].

*Corresponding author.
Email: qli289@aliyun.com (Q. Li), jbzmath@xznu.edu.cn (J. B. Zhang), dychen@mail.shu.edu.cn (D. Y. Chen)

Ablowitz-Ladik (A-L for short) spectral problem [28–30] attracts much attention. Usually, one eigenvalue problem admits a pair of linear problem, for which the hierarchy of soliton equation is just the compatible condition. For more details, one can see for example [31–35] and references therein. Recently, the discrete mKdV hierarchy with self-consistent sources is investigated by constructing non-auto-Bäcklund transformations [19], where the self-consistent sources are expressed by both of the eigenfunctions and adjoint eigenfunctions corresponding to the A-L spectral problem and its adjoint problem, respectively. However, there exists some relationship between the eigenfunctions and the adjoint eigenfunctions, which implies that the sources can be expressed only by the eigenfunctions. This relationship leads to another form of the discrete mKdV hierarchy with self-consistent sources (DmKdVSCS hierarchy), which is different from the hierarchy in [19,32].

The paper is organized as follows. In Section 2, we briefly deduce the discrete mKdV hierarchy with self-consistent sources. In Section 3, we describe the relationship between the eigenfunctions and the adjoint eigenfunctions, and deduce another form of the discrete mKdV hierarchy with self-consistent sources. In Section 4, we investigate the soliton solutions of DmKdVSCS hierarchy through the IST. Some dynamics are given.

2 The discrete mKdV hierarchy with self-consistent sources

In this section, we derive the discrete mKdV hierarchy with self-consistent sources (DmKdVSCS hierarchy). Our approach is a little different from the one using constrained flows [19,32].

We denote the function $f_n = f(n,t)$ with $n \in \mathbb{Z}$. The shift operator E is defined by $Ef_n = f_{n+1}$ for arbitrary function f_n. By Δ we denote $E-1$. The A-L spectral problem is known as [30]

$$\begin{pmatrix} \phi_{1,n+1} \\ \phi_{2,n+1} \end{pmatrix} = \begin{pmatrix} z & Q_n \\ R_n & 1/z \end{pmatrix} \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}. \tag{2.1}$$

Let us begin with A-L spectral problem in the case of $R_n = \varepsilon Q_n$ $(\varepsilon = \pm 1)$,

$$\begin{pmatrix} \phi_{1,n+1} \\ \phi_{2,n+1} \end{pmatrix} = \boldsymbol{M}_n \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}, \quad \boldsymbol{M}_n = \begin{pmatrix} z & Q_n \\ \varepsilon Q_n & 1/z \end{pmatrix}, \tag{2.2a}$$

and the time evolution

$$\begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}_t = \boldsymbol{U}_n \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}, \quad \boldsymbol{U}_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}. \tag{2.2b}$$

The adjoint spectral problem of (2.2a) is

$$\begin{pmatrix} \varphi_{1,n-1} \\ \varphi_{2,n-1} \end{pmatrix} = \boldsymbol{M}_n^T \begin{pmatrix} \varphi_{1,n} \\ \varphi_{2,n} \end{pmatrix}, \tag{2.3}$$

where the superscript T denotes the transpose of a matrix. From the compatibility condition, one have the zero curve equation

$$\boldsymbol{M}_{n,t}=(E\boldsymbol{U}_n)\boldsymbol{M}_n-\boldsymbol{M}_n\boldsymbol{U}_n. \tag{2.4}$$

Following that, one arrive

$$Q_{n,t}=Q_n(A_{n+1}-D_n)+\frac{1}{z}B_{n+1}-zB_n, \tag{2.5a}$$

$$\varepsilon Q_{n,t}=\varepsilon Q_n(D_{n+1}-A_n)+zC_{n+1}-\frac{1}{z}C_n, \tag{2.5b}$$

$$A_n=\frac{z_t}{z}n+\frac{1}{z}\Delta^{-1}(Q_nC_n-\varepsilon Q_nB_{n+1})+A^{(0)}, \tag{2.5c}$$

$$D_n=-\frac{z_t}{z}n+z\Delta^{-1}(B_n\varepsilon Q_n-Q_nC_{n+1})+D^{(0)}, \tag{2.5d}$$

where $A^{(0)}$ and $D^{(0)}$ are integration constants. Further one can get

$$\begin{pmatrix} Q_n \\ \varepsilon Q_n \end{pmatrix}_t=\Big(zL_1-\frac{1}{z}L_2\Big)\begin{pmatrix} B_n \\ C_n \end{pmatrix}+(A^{(0)}-D^{(0)})\begin{pmatrix} Q_n \\ -\varepsilon Q_n \end{pmatrix}, \tag{2.6}$$

where

$$L_1=\begin{pmatrix} -1 & 0 \\ 0 & E \end{pmatrix}+\begin{pmatrix} -Q_n \\ \varepsilon Q_nE \end{pmatrix}\Delta^{-1}(\varepsilon Q_n,-Q_nE), \tag{2.7a}$$

$$L_2=\begin{pmatrix} -E & 0 \\ 0 & 1 \end{pmatrix}-\begin{pmatrix} -Q_nE \\ \varepsilon Q_n \end{pmatrix}\Delta^{-1}(\varepsilon Q_nE,-Q_n). \tag{2.7b}$$

It is easy to verify that the inverse operators of L_1 and L_2 are

$$L_1^{-1}=\begin{pmatrix} -1 & 0 \\ 0 & E^{-1} \end{pmatrix}+\begin{pmatrix} Q_n \\ \varepsilon Q_{n-1} \end{pmatrix}\Delta^{-1}(\varepsilon Q_n,Q_n)\frac{1}{\mu_n}, \tag{2.8a}$$

$$L_2^{-1}=\begin{pmatrix} -E^{-1} & 0 \\ 0 & 1 \end{pmatrix}-\begin{pmatrix} Q_{n-1} \\ \varepsilon Q_n \end{pmatrix}\Delta^{-1}(\varepsilon Q_n,Q_n)\frac{1}{\mu_n}, \tag{2.8b}$$

where $\mu_n=1-\varepsilon Q_n^2$. To deduce the discrete mKdV hierarchy with self-consistent sources (DmKdVSCS hierarchy), one expands $(B_n,C_n)^T$ as

$$\begin{pmatrix} B_n \\ C_n \end{pmatrix}=\sum_{l=0}^{2k+1}\begin{pmatrix} b_{n,l} \\ c_{n,l} \end{pmatrix}z^{2k-2l+1}+\sum_{j=1}^{N}z\Big(\frac{\alpha_{n,j}}{z^2-z_j^2}+\frac{\varepsilon\beta_{n,j}}{z^2-z_j^{-2}}\Big), \tag{2.9}$$

where $\{z_j\}$ are N distinct eigenvalues satisfying spectral problem (2.2a), and the two-order column vectors $\{\alpha_{n,j}\}$ and $\{\beta_{n,j}\}$ satisfy

$$z_j^2L_1\alpha_{n,j}=L_2\alpha_{n,j},\quad L_1\beta_{n,j}=z_j^2L_2\beta_{n,j},\quad j=1,2,\cdots,N. \tag{2.10}$$

From the spectral problem (2.2a) and the adjoint spectral problem (2.3), by some direct computations, one have

$$\alpha_{n,j}=z_j\begin{pmatrix}\phi_{1,n,j}\varphi_{2,n-1,j}\\ \phi_{2,n,j}\varphi_{1,n-1,j}\end{pmatrix},\quad \beta_{n,j}=-z_j^{-1}\begin{pmatrix}\phi_{2,n,j}\varphi_{1,n-1,j}\\ \phi_{1,n,j}\varphi_{2,n-1,j}\end{pmatrix},\quad j=1,2,\cdots,N, \tag{2.11}$$

where $\phi_{n,j}=(\phi_{1,n,j},\phi_{2,n,j})^T$ and $\varphi_{n,j}=(\varphi_{1,n,j},\varphi_{2,n,j})^T$ satisfy (2.2a) and (2.3), respectively. Now we take $A^{(0)}=-D^{(0)}=(z+z^{-1})(z-z^{-1})^{2k+1}$, then (2.6) becomes

$$\begin{aligned}\begin{pmatrix}Q_n\\ \varepsilon Q_n\end{pmatrix}_t=&\sum_{l=0}^{2k+1}z^{2k-2l}(z^2L_1+L_2)\begin{pmatrix}b_{n,l}\\ c_{n,l}\end{pmatrix}\\ &+\sum_{m=0}^{2k+1}(-1)^mC_{2k+1}^m z^{2k-2m}(z^2+1)\cdot\begin{pmatrix}Q_n\\ -\varepsilon Q_n\end{pmatrix}+\sum_{j=1}^N L_1(\alpha_{n,j}+\varepsilon\beta_{n,j}),\end{aligned} \tag{2.12}$$

where C_{2k+1}^m is for combination. By comparing the coefficients of the same power of z in (2.12), we obtain the discrete mKdV hierarchy with self-consistent sources (DmKdVSCS hierarchy)

$$\begin{aligned}\begin{pmatrix}Q_n\\ \varepsilon Q_n\end{pmatrix}_t=&(L-2I+L^{-1})^k\mu_n\begin{pmatrix}Q_{n+1}-Q_{n-1}\\ \varepsilon(Q_{n+1}-Q_{n-1})\end{pmatrix}\\ &+\mu_n\sum_{j=1}^N\begin{pmatrix}-\phi_{1,n,j}\varphi_{2,n,j}+\varepsilon\phi_{2,n,j}\varphi_{1,n,j}\\ \phi_{2,n,j}\varphi_{1,n,j}-\varepsilon\phi_{1,n,j}\varphi_{2,n,j}\end{pmatrix},\quad k=0,1,2,\cdots,\end{aligned} \tag{2.13}$$

where

$$\phi_{1,n+1,j}(z_j)=z_j\phi_{1,n,j}(z_j)+Q_n\phi_{2,n,j}(z_j), \tag{2.14a}$$
$$\phi_{2,n+1,j}(z_j)=\varepsilon Q_n\phi_{1,n,j}(z_j)+z_j^{-1}\phi_{2,n,j}(z_j), \tag{2.14b}$$
$$\varphi_{1,n-1,j}(z_j)=z_j\varphi_{1,n,j}(z_j)+\varepsilon Q_n\varphi_{2,n,j}(z_j), \tag{2.14c}$$
$$\varphi_{2,n-1,j}(z_j)=Q_n\varphi_{1,n,j}(z_j)+z_j^{-1}\varphi_{2,n,j}(z_j),\quad j=1,2,\cdots,N, \tag{2.14d}$$

and $L=L_2L_1^{-1}$, I is an unit matrix. The corresponding Lax pair (2.2) is governed by,

$$\begin{aligned}\begin{pmatrix}B_n\\ C_n\end{pmatrix}=&\left(-\sum_{l=0}^{k}\sum_{m=1}^{l+l}\Upsilon_mL_1^{-1}L^{l-m+1}+\sum_{l=k+1}^{2k+1}\sum_{m=l+2}^{2k+3}\Upsilon_mL_2^{-1}L^{l-m+2}\right)\begin{pmatrix}Q_n\\ -R_n\end{pmatrix}\\ &+\sum_{j=1}^N\left[\frac{z_jz}{z^2-z_j^2}\begin{pmatrix}\phi_{1,n,j}\varphi_{2,n-1,j}\\ \phi_{2,n,j}\varphi_{1,n-1,j}\end{pmatrix}-\frac{\varepsilon z_j^{-1}z}{z^2-z_j^{-2}}\begin{pmatrix}\phi_{2,n,j}\varphi_{1,n-1,j}\\ \phi_{1,n,j}\varphi_{2,n-1,j}\end{pmatrix}\right],\end{aligned} \tag{2.15a}$$

$$\begin{aligned}A_n=&\frac{1}{2}(z+z^{-1})(z-z^{-1})^{2k+1}+\frac{1}{2}\Delta^{-1}\Big[z^{-1}(-\varepsilon Q_nE,Q_n)\\ &+z(-\varepsilon Q_n,Q_nE)\Big]\begin{pmatrix}B_n\\ C_n\end{pmatrix},\end{aligned} \tag{2.15b}$$

where $\{Y_i\}$ are defined as $Y_l=(-1)^{l-1}(C_{2k+1}^{l-1}-C_{2k+1}^{l-2})$, $Y_1=1$, $Y_{2k+3}=-1$ $(l=2,3,\cdots,2k+2)$.
Specially, taking $k=0$, Eq. (2.13) reads

$$\left(\begin{array}{c} Q_n \\ \varepsilon Q_n \end{array}\right)_t=\mu_n\left[\left(\begin{array}{c} Q_{n+1}-Q_{n-1} \\ \varepsilon(Q_{n+1}-Q_{n-1}) \end{array}\right)+\sum_{j=1}^{N}\left(\begin{array}{c} -\phi_{1,n,j}\varphi_{2,n,j}+\varepsilon\phi_{2,n,j}\varphi_{1,n,j} \\ \phi_{2,n,j}\varphi_{1,n,j}-\varepsilon\phi_{1,n,j}\varphi_{2,n,j} \end{array}\right)\right], \tag{2.16}$$

of which first row provides the discrete mKdV equation with self-consistent sources (DmKdVSCS equation)

$$Q_{n,t}=(1-\varepsilon Q_n^2)\Big[(Q_{n+1}-Q_{n-1})+\sum_{j=1}^{N}(-\phi_{1,n,j}\varphi_{2,n,j}+\varepsilon\phi_{2,n,j}\varphi_{1,n,j})\Big]. \tag{2.17}$$

3 The relationship between the eigenfunctions and the adjoint ones

As in the previous section, the term of sources in Eq. (2.13) is presented by both of the eigenfunctions and adjoint eigenfunctions corresponding to the spectral problem (2.2a) and the adjoint spectral problem (2.3) respectively. Based on this, we would like to discuss another representation for the self-consistent sources in this section. We shall show that only the eigenfunctions appear in the self-consistent sources instead of the adjoint eigenfunctions. In fact, there exists some relationship between the eigenfunctions and the adjoint ones. In order to verify this fact, we give two lemmas firstly.

Lemma 3.1. *Suppose* $\phi_n=(\phi_{1,n},\phi_{2,n})^T$ *and* $\varphi_n=(\varphi_{1,n},\varphi_{2,n})^T$ *satisfy* (2.2a) *and* (2.3), *respectively. Then the formulae holds,*

$$\phi_{1,n+1}\varphi_{1,n}=-\phi_{2,n+1}\varphi_{2,n}, \tag{3.1}$$

where the boundary condition is $\phi_{1,1}\varphi_{1,0}+\phi_{2,1}\varphi_{2,0}=0$.

Proof. Multiply the first row of (2.2a) by $\varphi_{1,n}$, the second row of (2.2a) by $\varphi_{2,n}$, and plus them, one have

$$\phi_{1,n+1}\varphi_{1,n}+\phi_{2,n+1}\varphi_{2,n}=\phi_{1,n}(z\varphi_{1,n}+\varepsilon Q_n\varphi_{2,n})+\phi_{2,n}(Q_n\varphi_{1,n}+z^{-1}\varphi_{2,n}), \tag{3.2}$$

which can reach (3.1) by making use of (2.2a). □

Lemma 3.2. *Suppose that* $\phi_n=(\phi_{1,n},\phi_{2,n})^T$ *satisfy* (2.2a) *and denote that* $S_n=\Pi_{j=n}^{\infty}(1-\varepsilon Q_j^2)$. *Let*

$$\varphi_{1,n}=S_{n+1}\phi_{2,n+1}, \quad \varphi_{2,n}=-S_{n+1}\phi_{1,n+1}. \tag{3.3}$$

Then $\varphi_n=(\varphi_{1,n},\varphi_{2,n})^T$ *satisfy* (2.3).

Proof. By straightforward calculation. Noting that $S_n=\mu_n S_{n+1}$. □

Then making use of the two lemmas, we arrived the theorem as follows.

Theorem 3.1. *Suppose that* $\phi_n=(\phi_{1,n},\phi_{2,n})^T$ *satisfy* (2.2a) *and denote that* $S_n=\Pi_{j=n}^{\infty}(1-\varepsilon Q_j^2)$. *The discrete mKdV hierarchy with self-consistent sources (DmKdVSCS hierarchy)* (2.13) *is rewritten as*

$$\begin{aligned}\begin{pmatrix} Q_n \\ \varepsilon Q_n \end{pmatrix}_t =&(L-2I+L^{-1})^k\mu_n\begin{pmatrix} Q_{n+1}-Q_{n-1} \\ \varepsilon(Q_{n+1}-Q_{n-1}) \end{pmatrix}\\ &+S_n\sum_{j=1}^{N}\begin{pmatrix} \phi_{1,n,j}\phi_{1,n+1,j}+\varepsilon\phi_{2,n,j}\phi_{2,n+1,j} \\ \phi_{2,n,j}\phi_{2,n+1,j}+\varepsilon\phi_{1,n,j}\phi_{1,n+1,j} \end{pmatrix}, \quad k=0,1,2,\cdots,\end{aligned} \tag{3.4}$$

with

$$\phi_{1,n+1,j}(z_j)=z_j\phi_{1,n,j}(z_j)+Q_n\phi_{2,n,j}(z_j), \tag{3.5a}$$

$$\phi_{2,n+1,j}(z_j)=\varepsilon Q_n\phi_{1,n,j}(z_j)+z_j^{-1}\phi_{2,n,j}(z_j), \quad j=1,2,\cdots,N. \tag{3.5b}$$

Thus, Eqs. (2.15) are rewritten as

$$\begin{aligned}\begin{pmatrix} B_n \\ C_n \end{pmatrix}=&\Bigg(-\sum_{l=0}^{k}\sum_{m=1}^{l+l}\Upsilon_m L_1^{-1}L^{l-m+1}+\sum_{l=k+1}^{2k+1}\sum_{m=l+2}^{2k+3}\Upsilon_m L_2^{-1}L^{l-m+2}\Bigg)\begin{pmatrix} Q_n \\ -R_n \end{pmatrix}\\ &+S_n\sum_{j=1}^{N}\left[\frac{z_jz}{z^2-z_j^2}\begin{pmatrix} -\phi_{1,n,j}^2 \\ \phi_{2,n,j}^2 \end{pmatrix}+\frac{\varepsilon z_j^{-1}z}{z^2-z_j^{-2}}\begin{pmatrix} -\phi_{2,n,j}^2 \\ \phi_{1,n,j}^2 \end{pmatrix}\right],\end{aligned} \tag{3.6a}$$

$$\begin{aligned}A_n=&\frac{1}{2}(z+z^{-1})(z-z^{-1})^{2k+1}+\frac{1}{2}\Delta^{-1}\Big[z^{-1}(-\varepsilon Q_nE,Q_n)\\ &+z(-\varepsilon Q_n,Q_nE)\Big]\begin{pmatrix} B_n \\ C_n \end{pmatrix}.\end{aligned} \tag{3.6b}$$

The DmKdVSCS equation (2.17) is rewritten as

$$Q_{n,t}=(1-\varepsilon Q_n^2)(Q_{n+1}-Q_{n-1})+S_n\sum_{j=1}^{N}\Big[\phi_{1,n,j}\phi_{1,n+1,j}+\varepsilon\phi_{2,n,j}\phi_{2,n+1,j}\Big]. \tag{3.7}$$

In the later section, we investigate these equations.

4 Multi-soliton solutions to the DmKdVSCS hierarchy

In this section, we solve the DmKdVSCS hierarchy (3.4) through the IST. Since DmKdVSCS hierarchy are deduced in the reduction case of $R_n=\varepsilon Q_n$, the Ablowitz-Ladik hierarchies and the DmKdVSCS hierarchy share the same direct scattering problem. One can refer to [30] for the IST of the Ablowitz-Ladik spectral problem. Thus, we shall only list main results of the direct scattering problem (cf. [30, 36]). Then we give the inverse scattering problem of the DmKdVSCS hierarchy.

4.1 The direct scattering problem

Assume Q_n and R_n decrease rapidly when $|n|$ tends to infinity. There are four discrete Jost functions $\psi_n^+(z)$, $\bar{\psi}_n^+(z)$, $\psi_n^-(z)$, $\bar{\psi}_n^-(z)$, which have the asymptotic behaviors

$$\psi_n^+(z)\sim\begin{pmatrix}1\\0\end{pmatrix}z^n,\qquad \bar{\psi}_n^+(z)\sim\begin{pmatrix}0\\1\end{pmatrix}z^{-n},\qquad n\to+\infty, \tag{4.1a}$$

$$\psi_n^-(z)\sim\begin{pmatrix}0\\-1\end{pmatrix}z^{-n},\qquad \bar{\psi}_n^-(z)\sim\begin{pmatrix}1\\0\end{pmatrix}z^{n},\qquad n\to-\infty. \tag{4.1b}$$

$\psi_n^+(z)$ and $\psi_n^-(z)$ are analytic for $|z|\le 1$, $\bar{\psi}_n^+(z)$ and $\bar{\psi}_n^-(z)$ are analytic for $|z|>1$. On the unit circle $|z|=1$, there are the following relations from (4.1)

$$\begin{pmatrix}\bar{\psi}_{1,n}^+(z)\\ \bar{\psi}_{2,n}^+(z)\end{pmatrix}=\begin{pmatrix}\psi_{2,n}^{+*}(z)\\ \psi_{1,n}^{+*}(z)\end{pmatrix},\quad \begin{pmatrix}\bar{\psi}_{1,n}^-(z)\\ \bar{\psi}_{2,n}^-(z)\end{pmatrix}=-\begin{pmatrix}\psi_{2,n}^{-*}(z)\\ \psi_{1,n}^{-*}(z)\end{pmatrix}. \tag{4.2}$$

Setting

$$\psi_n^-(z)=a(z)\bar{\psi}_n^+(z)+b(z)\psi_n^+(z), \tag{4.3a}$$
$$\bar{\psi}_n^-(z)=-\bar{a}(z)\psi_n^+(z)+\bar{b}(z)\bar{\psi}_n^+(z), \tag{4.3b}$$

yields

$$a(z)=W_n(\psi_n^+(z),\psi_n^-(z)),\qquad b(z)=W_n(\psi_n^-(z),\bar{\psi}_n^+(z)), \tag{4.4a}$$
$$\bar{a}(z)=-W_n(\bar{\psi}_n^+(z),\bar{\psi}_n^-(z)),\qquad \bar{b}(z)=W_n(\bar{\psi}_n^-(z),\psi_n^+(z)), \tag{4.4b}$$

where $W_n(f_n(z),g_n(z))$ is the Wronskian of $f_n(z)$ and $g_n(z)$ defined by

$$W_n(f_n(z),g_n(z))=\begin{vmatrix}f_{1,n}(z) & g_{1,n}(z)\\ f_{2,n}(z) & g_{2,n}(z)\end{vmatrix}=f_{1,n}(z)g_{2,n}(z)-f_{2,n}(z)g_{1,n}(z). \tag{4.5}$$

$a(z)$ is analytic for $|z|\le 1$ and $\bar{a}(z)$ is analytic for $|z|>1$. On the unit circle $|z|=1$, there are

$$a(z)\bar{a}(z)+b(z)\bar{b}(z)=1. \tag{4.6}$$

Function $a(z)$ has only a finite number of zeros at $z_1,z_2,\cdots,z_N$ in the region $|z|<1$ and function $\bar{a}(z)$ at $\bar{z}_1,\bar{z}_2,\cdots,\bar{z}_{\bar{N}}$ in the region $|z|>1$, which are all the discrete eigenvalues of the spectral problem (2.2). At those zeros, $\psi_n^+(z)$, $\bar{\psi}_n^+(z)$, $\psi_n^-(z)$ and $\bar{\psi}_n^-(z)$ have the linear relation

$$\psi_n^-(z_j)=p_j\psi_n^+(z_j),\quad j=1,2,\cdots,N,\quad \bar{\psi}_n^-(\bar{z}_j)=\bar{p}_j\bar{\psi}_n^+(\bar{z}_j),\quad j=1,2,\cdots,\bar{N}, \tag{4.7}$$

where $p_j\doteq p(z_j)$, $\bar{p}_j\doteq\bar{p}(\bar{z}_j)$. From (2.2) and (4.7), one have

$$2\sum_{n=-\infty}^{\infty}c_j^2S_n\psi_{1,n}^+(z_j)\psi_{2,n}^+(z_j)=1,\qquad j=1,2,\cdots,N, \tag{4.8a}$$

$$2\sum_{n=-\infty}^{\infty}\bar{c}_j^2S_n\bar{\psi}_{1,n}^+(z_j)\bar{\psi}_{2,n}^+(z_j)=1,\qquad j=1,2,\cdots,\bar{N}, \tag{4.8b}$$

where $c_j^2 = p_j/z_j\dot{a}(z_j)$, $j=1,2,\cdots,N$, $\bar{c}_j^2 = \bar{p}_j/\bar{z}_j\dot{\bar{a}}(\bar{z}_j)$, $j=1,2,\cdots,\bar{N}$. Here a dot stands for differentiation with respect to z. $c_j \doteq c_j(t)$ and $\bar{c}_j \doteq \bar{c}_j(t)$ are so called normalization constants. $c_j\psi_n^+(z_j)$ and $\bar{c}_j\bar{\psi}_n^+(\bar{z}_j)$ are the normalization eigenfunctions. The set

$$\Big\{R(z)=\frac{b(z)}{a(z)},\, c_j,\, z_j(|z_j|\le 1,\, j=1,2,\cdots,N);\, \bar{R}(z)=\frac{\bar{b}(z)}{\bar{a}(z)},\, \bar{c}_j,\, \bar{z}_j(|\bar{z}_j|>1,\, j=1,2,\cdots,\bar{N})\Big\} \quad (4.9)$$

is just the scattering data of the spectral problem (2.1).

4.2 The time evolutions of the scattering data

In the case of $R_n = \varepsilon Q_n$, by $\psi_n(t,z_m)$ denote the normalized Jost solution of $\psi_n^+(t,z)$ with $z=z_m$ and normalization constant $c_m(t)$, i.e.,

$$\psi_n(t,z_m) = c_m(t)\psi_n^+(t,z_m), \quad m=1,2,\cdots,N. \quad (4.10)$$

Assume that

$$\phi_{n,j} = \sqrt{2\beta_j(t)}\psi_n(t,z_j), \quad j=1,\cdots,N, \quad (4.11)$$

where $\{\phi_{n,j}\}$ satisfy the spectral problem (2.2) with $z=z_j$, $\{\beta_j(t)\}$ is arbitrary continuous functions of t. It follows that

$$\beta_j(t) = \sum_{n=-\infty}^{\infty} c_j^2(t)\phi_{1,n,j}\phi_{2,n,j}, \quad j=1,2,\cdots,N. \quad (4.12)$$

It is found from (2.2) and (4.11) that

$$\lim_{z\to z_m}\sum_{j=1}^{2N}\frac{z_j z}{z^2-z_j^2}\sum_{n=-\infty}^{+\infty} S_n(\phi_{1,n,j}^2\psi_{2,n}^2(z)-\phi_{2,n,j}^2\psi_{1,n}^2(z))=0, \quad m=1,2,\cdots,N, \quad (4.13a)$$

$$\lim_{z\to z_m}(A_n\psi_{1,n}(z)+B_n\psi_{2,n}(z)) \sim z_m^n c_m(t)\beta_m(t), \quad n\to+\infty, \quad m=1,2,\cdots,N, \quad (4.13b)$$

where A_n, B_n and C_n are governed by (3.6). Thus, we can determine the time evolutions of the scattering data (4.9) as follows.

Theorem 4.1. *The discrete scattering data* (4.9) *satisfy* ($k=0,1,2,\cdots,$)

$$c_j(t) = c_j(0)\exp\Big[\frac{1}{2}(z_j-z_j^{-1})^{2k+1}(z_j+z_j^{-1})t+\int_0^t\beta_j(\tau)d\tau\Big], \quad j=1,2,\cdots,N, \quad (4.14)$$

where $\{z_j\}$ *are constants,* $c_j(0)$ *are the initial value of* $c_j(t)$.

4.3 Exact solutions

Following the standard IST procedure (cf. [30]), the N-soliton solutions for the DmKd-VSCS hierarchy (2.13) are derived in the following way

$$Q_n = -\Lambda^T_{n+1}(t)W_n^{-1}(t)\Lambda_n(t), \tag{4.15a}$$

$$\begin{aligned}\phi_{1,n,j}(t) = &\sqrt{2\beta_j(t)}c_j(t)\sum_{l=0}^{\infty}\Big[\sum_{s=0}^{\infty}\Lambda^T_{n+2s+1}(t)W_n^{-1}(t)\Lambda_n(t)\sum_{m=1}^{N}\varepsilon c_m^2(t)z_m^{2n+2s+2l+1}\\ &-\delta_{0,l}\Big]z_j^{n+2l},\end{aligned} \tag{4.15b}$$

$$\phi_{2,n,j}(t) = \varepsilon\sqrt{2\beta_j(t)}c_j(t)\sum_{l=0}^{\infty}\Lambda^T_{n+2l+1}(t)V_n^{-1}(t)\Lambda_n(t)z_j^{n+2l+1}, \quad j=1,2,\cdots,N, \tag{4.15c}$$

where

$$c_j(t) = c_j(0)\exp\Big[\frac{1}{2}(z_j - z_j^{-1})^{2k}(z_j^2 - z_j^{-2})t + \int_0^t \beta_j(\tau)d\tau\Big], \quad j=1,2,\cdots,N, \tag{4.16a}$$

$$\Lambda_n(t) = (c_1(t)z_1^n, c_2(t)z_2^n, \cdots, c_N(t)z_N^n)^T, \quad W_n(t) = I - E^T_{n+1}E_n, \tag{4.16b}$$

$$V_n(t) = I - E_nE^T_{n+1}, \quad E_n = (e_{mj})_{N\times N}, \quad e_{mj} = \begin{cases} \dfrac{c_j(t)c_m(t)z_j^{n+1}}{(z_m^{-2}-z_j^2)z_m^{-n+1}}, & \varepsilon = 1, \\ \dfrac{ic_j(t)c_m(t)z_j^{n+1}}{(z_m^{-2}-z_j^2)z_m^{-n+1}}, & \varepsilon = -1. \end{cases} \tag{4.16c}$$

That is, for different k, we can recover $z_j(t)$ and $c_j(t)$ from their initial value $z_j(0)$ and $c_j(0)$, and further get solutions for the whole DmKdVSCS hierarchy (3.4).

Specially, the one-soliton for the DmKdVSCS equation (2.17) is (corresponding to $k=0$, $N=1$),

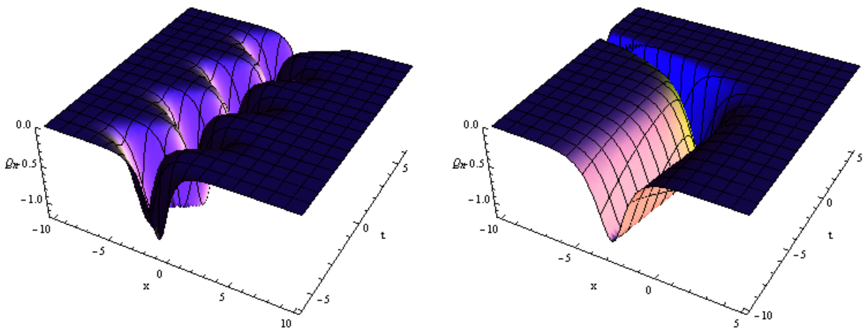
$$Q_n = -\frac{c_1^2(t)z_1^{2n+1}}{1 - \frac{\varepsilon c_1^4(t)z_1^{4n+2}}{(z_1^{-2}-z_1^2)^2}}, \tag{4.17a}$$

$$\phi_{1,n,1}(t) = \sqrt{2\beta_1(t)}c_1(t)\Big[-z_1^n + \frac{\varepsilon c_1^4(t)z_1^{5n-2}}{(z_1^{-2}-z_1^2)^2 - \varepsilon c_1^4(t)z_1^{4n+2}}\Big], \tag{4.17b}$$

$$\phi_{2,n,1}(t) = \sqrt{2\beta_1(t)}\frac{\varepsilon c_1^3(t)z_1^n(z_1^{-2}-z_1^2)}{(z_1^{-2}-z_1^2)^2 - \varepsilon c_1^4(t)z_1^{4n+2}}, \tag{4.17c}$$

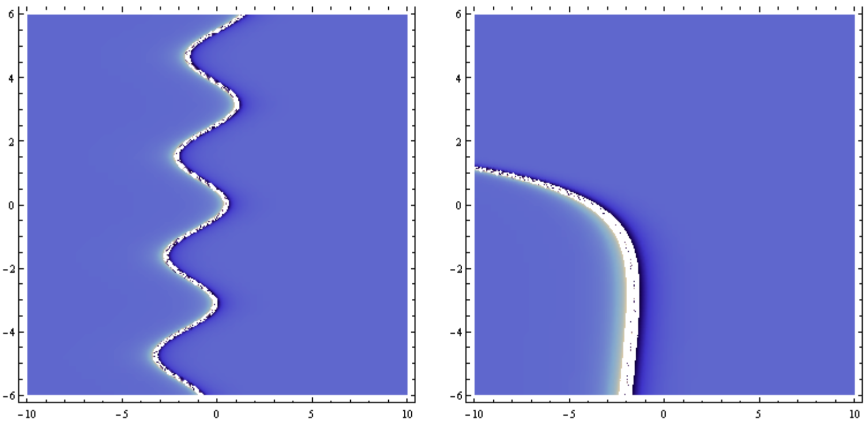
where $c_1(t) = c_1(0)e^{\frac{1}{2}(z_1^2 - z_1^{-2})t + \int_0^t \beta_1(\tau)d\tau}$, $c_1(0)$, z_1 are constants, and $\beta_1(t)$ is an arbitrary function of t. In the case of $\varepsilon = -1$, (4.17a) reads

$$Q_n = \frac{1}{2}(z_1^2 - z_1^{-2})\mathrm{sech}\xi, \quad \xi = (z_1^{-2} - z_1^2)t + n\ln z_1^2 + \int_0^t \beta_1(\tau)d\tau + \xi_0, \tag{4.18}$$



(a) (b)

Figure 1: The shape and motion of one soliton of DmKdVSCS equation (2.17) in the case of $\varepsilon=-1$. (a) The plot of the one-soliton solution given by (4.18) for $\beta_1(t)=2-2\sin 2t$, $z_1=0.5$, $\xi_0=0$, and $n\in[-10,10]$, $t\in[-6,6]$. (b) The plot of the one-soliton solution given by (4.18) for $\beta_1(t)=2-2e^t$, $z_1=0.5$, $\xi_0=0$, and $n\in[-10,5]$, $t\in[-10,5]$.



(a) (b)

Figure 2: The density plot of a moving soliton of DmKdVSCS equation (2.17) in the case of $\varepsilon=1$. (a) The plot of the one-soliton solution given by (4.19) for $\beta_1(t)=2-2\sin 2t$, $z_1=0.5$, $\xi_0=0$, and $n\in[-10,10]$, $t\in[-6,6]$. (b) The plot of the one-soliton solution given by (4.19) for $\beta_1(t)=2-2e^t$, $z_1=0.5$, $\xi_0=0$, and $n\in[-10,10]$, $t\in[-6,6]$.

and in the case of $\varepsilon=1$, (4.17a) reads

$$Q_n=\frac{1}{2}(z_1^{-2}-z_1^2)\operatorname{csch}\xi, \tag{4.19}$$

where ξ_0 is constant. The arbitrary continuous function $\beta_1(t)$ changes the velocity of the soliton but not the shape. A variety of travelling trajectories can be derived by choosing different $\beta_1(t)$. Fig. 1 and Fig. 2 describe the shape and motion of one soliton for the DmKdVSCS equation (2.17).

5 Conclusions

In the paper, starting from the Ablowitz-Ladik spectral problem, we deduce the DmKdVSCS hierarchy. The sources are expressed only by the eigenfunctions of A-L spectral problem instead of the adjoint spectral problem. By the inverse scattering transform we derived the multi-soliton solutions of the hierarchy.

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