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A Weak Formulation for Solving the Elliptic Interface Problems with Imperfect Contact

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Abstract. We propose a non-traditional finite element method with non-body-fitting grids to solve the matrix coefficient elliptic equations with imperfect contact in two dimensions, which has not been well-studied in the literature. Numerical experiments demonstrated the effectiveness of our method.

AMS subject classifications: 65N30

Key words: Elliptic equation, jump condition, matrix coefficient, imperfect contact.

1 Introduction and formulations

Elliptic interface problems occur in a variety of disciplines when there are multi-physics and multi-phase materials, such as in electromagnetics, material science, and fluid dynamics. In this paper we consider a class of special elliptic interface problems with imperfect contact conditions, which occurs in heat transfer in composite media, heat transfer in building [1], transient behavior for the thermoelastic contact of two rods of dissimilar materials [2], etc.

The numerical model of the elliptic interface problem with imperfect contact is as follows: for ease of discussion, we consider a rectangular domain $\Omega=(x_{\min},x_{\max})\times(y_{\min},y_{\max})$ (If the boundary is of general geometry, we can imbed the research domain in a larger rectangular box. Please refer to [3] for details on dealing with boundaries with general geometry). Γ is an interface prescribed by the zero level-set $\{(x,y)\in\Omega \mid \phi(x,y)=$

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$0\}$ of a level-set function $\phi(x,y)$. The advantage of using the level-set function is to represent interface cut locations on the grids without having to parameterize the interface. The unit normal vector of Γ is $\mathbf{n}=\frac{\nabla\phi}{|\nabla\phi|}$ pointing from $\Omega^-=\{(x,y)\in\Omega \mid \phi(x,y)\le 0\}$ to $\Omega^+=\{(x,y)\in\Omega \mid \phi(x,y)\ge 0\}$. Consider the problem with imperfect contact condition:

$$\begin{aligned}
&-\nabla\cdot(\beta(x)\nabla u(x))=f(x), && x\in\Omega^{\pm},\\
&[u(x)]=\lambda(x)\beta^+(x)\nabla u^+(x)\cdot\mathbf{n}, && x\in\Gamma,\\
&[(\beta(x)\nabla u(x))\cdot\mathbf{n}]=0, && x\in\Gamma,\\
&u(x)=g(x), && x\in\partial\Omega,
\end{aligned}$$

in which $x=(x_1,x_2)$ denotes the spatial variables and ∇ is the gradient operator. The coefficient $\beta(x)$ is assumed to be a 2×2 matrix that is uniformly elliptic on each disjoint subdomain, Ω^- and Ω^+, and its components are continuously differentiable on each disjoint subdomain, but they may be discontinuous across the interface Γ. The right-hand side $f(x)$ is assumed to lie in $L^2(\Omega)$. The bracket $[\ ]$ means the jump, i.e., the difference from two sides of the interface.

We introduce the weak solution by the standard procedure of multiplying by a test function and integrating by parts. For the problem with Dirichlet boundary condition, we have

$$\int_{\Omega^+}\beta\nabla u\cdot\nabla\psi+\int_{\Omega^-}\beta\nabla u\cdot\nabla\psi=\int_{\Omega}f\psi, \tag{1.1}$$

where ψ is in H_0^1. The traditional finite element method has both the test and trial functions using the same basis. We will discuss later that our method does not use the same basis.

Here we briefly summarize the history of interface problems. The pioneering work on interface problems was the immersed boundary method [4,5] by Peskin in 1977. It uses a numerical approximation of the δ-function, which smears out the solution in a narrow band around the interface Γ. In [6], the immersed boundary method was combined with the level set method, resulting in a first order numerical method that is simple to implement, even in multiple spatial dimensions. However, for both methods, the numerical smearing at the interface forces continuity of the solution at the interface, regardless of the interface condition $[u]=a$, where a might not be zero.

The immersed interface method presented in [7] is a finite difference method with second-order accuracy. This method incorporates the interface conditions into the finite difference stencil, provided that neither of the two jump conditions are zero. The corresponding linear system is sparse, but not symmetric or positive definite. Various applications and extensions of the immersed interface method are discussed in [8].

In [9], on basis of the "immersed interface" method, a fast iterative method was proposed to solve constant coefficient problems with the interface conditions $[u]=0$ and $[\beta u_{\mathbf{n}}]\neq 0$. Non-body-fitting Cartesian grids are used, and then associated uniform triangulations are added on. Interfaces are not necessarily aligned with cell boundaries.