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Two New Energy-Preserving Algorithms for Generalized Fifth-Order KdV Equation

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Abstract. In this paper, based on the multi-symplectic formulations of the generalized fifth-order KdV equation and the averaged vector field method, two new energy-preserving methods are proposed, including a new local energy-preserving algorithm which is independent of the boundary conditions and a new global energy-preserving method. We prove that the proposed methods preserve the energy conservation laws exactly. Numerical experiments are carried out, which demonstrate that the numerical methods proposed in the paper preserve energy well.

AMS subject classifications: 65M10, 78A48

Key words: Generalized fifth-order KdV equation, local energy-preserving, global energy-preserving, average vector field, Fourier pseudospectral.

1 Introduction

The generalized fifth-order KdV equation, also known as generalized Kawahara equation, is a typical model equation for phasma waves, capillary-gravity water waves, as well as other dispersive phenomena when the dispersion in the cubic KdV-type equation is weak. The equation can be written in the general form

$$2u_t+\alpha u_{xxx}+\beta u_{xxxxx}=\partial_x f(u,u_x,u_{xx}),\quad (x,t)\in[-L,L]\times(0,T], \tag{1.1}$$

where α and β are real parameters with $\beta\neq 0$ and $f(u,u_x,u_{xx})$ is a smooth function. The fifth-order term is called the Kawahara term, and its coefficient β is known as the dispersion coefficient. Due to Kawahara term, it is very difficult to compute the solutions

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of these equations accurately and efficiently. The difficulty, associated with the fifth-order term, is due to fact that for $\alpha<0$ solutions of these equations exhibit highly oscillatory behaviors. This equation was first presented by Kawahara in 1972. Much attention has been paid to the Kwahara-type equations in the last few years. Various numerical methods have been developed for this type of equations, including the sinc method and the decomposition series method [22], the Dual-Petrov-Galerkin method [29], the multi-symplectic Preissmann method [21], the semi-explicit finite differnece scheme [7], the meshless method [1] and so on.

The conservation of energy is a crucial property for the generalized fifth-order KdV equation and Eq. (1.1) has the following energy conservation law

$$\int\Big(\frac{1}{2}\alpha v^2+F+\frac{1}{2\beta}(E^2-q^2)\Big)dx,$$

where

$$f(u,v,s)=F_u(u,v)-vF_{uv}(u,v)-sF_{vv}(u,v)+2sE_u(u,v)+svE_{uv}(u,v)+v^2E_{uu}(u,v).$$

Therefore, it is natural to require a discretization to reflect this property. However, there has been few approaches which can preserve the energy of the original equation in the literature. This motivates our study to introduce energy-preserving methods for solving generalized fifth-order KdV equation. Energy-preserving method is a geometric method that can preserve one or more physical/geometric properties of the system exactly. Feng [14, 15] first presented the concept of symplectic schemes for Hamiltonian systems and further the structure-preserving algorithms for the general conservative dynamical systems. Afterwards, Marsden et al. [23], Bridges [2] and Reich [3] introduced the concept of multi-symplectic integrators based on a multi-symplectic structure of some conservative PDEs. In some fields, it is more convenient to construct numerical algorithms that preserve the energy conservation law rather than the symplectic or multi-symplectic ones [20].

Nowadays, energy-preserving methods have been successfully applied to various aspects on numerical PDEs [4,10–13,16,26]. Furihata [17] presented the discrete variational derivative methods for a large class of PDEs that inherit energy conservation or dissipation property. Matsuo and Furihata [24] generalized the discrete variational derivative methods for complex-valued nonlinear PDEs. Recently, Celledoni et al. [8] used the averaged vector field (AVF) method to construct a class of systematic energy-preserving methods based on symplectic formulation of Hamiltonian PDEs.

The AVF method appeared firstly in [25]. It is identified as energy-preserving and as a B-series method in [9]. For ordinary differential equation

$$\dot{y}=f(y),\quad y\in\mathbb{R}^d, \tag{1.2}$$

the AVF method is the map $y\longmapsto y'$ defined by

$$\frac{y'-y}{\tau}=\int_0^1 f((1-\xi)y+\xi y')d\xi, \tag{1.3}$$

where τ is the time step. The AVF method exactly preserves the energy of Hamiltonian systems, and in contrast to projection-type integrators, only requires evalutions of the vector field. For polynomial Hamiltonians, the integral can be evaluated exactly, and the implementation is comparable to that of the implicit mid-point rule [19].

However, almost all existing energy-preserving methods only preserve the global energy and could only be applied to some PDEs with suitable boundary conditions, such as periodic or homogeneous boundary conditions. In 2008, Wang et al. [28] presented the concept of local structure-preserving algorithms (SPAs) for PDEs. The applicability of the local structure-preserving algorithm is irrespective of boundary conditions. Furthermore, the local SPAs for problems with proper boundary conditions are global SPAs, but the inverse is not necessarily valid. They developed the concatenating method and constructed several local SPAs for the Klein-Gordon equation. Cai et al. [5,6] used the concatenating method to construct local energy-preserving and momentum-preserving algorithms for the coupled nonlinear Schrödinger equation and the "good" Boussinesq equation. In [18], Gong et al. started from general multi-symplectic formulations of Hamiltonian PDEs and utilized the AVF method to construct a local energy-preserving (LEP) method, a class of global energy-preserving (GEP) methods and a local momentum-preserving method.

In this paper, we start from the multi-symplectic formulations of the generalized fifth-order KdV equation. Then we apply the AVF method in the time direction and use the mid-point rule and Fourier pseudospectral method in space direction discretization respectively. By this procedure, a local energy-preserving method and a global energy-preserving method are obtained respectively.

The outline of this article is as follows. In Section 2, conservation laws of the generalized fifth-order KdV equation are briefly introduced. Then the definitions and main properties of some operators are given in Section 3. Two new energy-preserving methods are proposed for the generalized fifth-order KdV equation in Section 4. Linear stability of the derived method is investigated in Section 5. Numerical experiments are carried out in Section 6, showing the good numerical performance of the two new schemes. Finally some conclusions are given in the last section.

2 Conservation laws of generalized fifth-order KdV equation

We can write the generalized fifth-order KdV equation in the multi-symplectic Hamiltonian system

$$M\partial_t z + K\partial_x z = \nabla_z S(z), \quad z \in \mathbb{R}^6, \tag{2.1}$$

where

$$z = (\psi, u, v, w, p, q)^T$$

is the state variable, the skew-symmetric matrices are

$$M=\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad K=\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and the Hamiltonian function is

$$S(z)=\frac{1}{2}\alpha v^2+\frac{1}{2\beta}q^2+wu+pv+F(u,v)+\frac{1}{2\beta}(2q+E(u,v))E(u,v). \tag{2.2}$$

The multi-symplectic reformulation of generalized fifth-order KdV equation is interesting for several reasons, since it has three local conservation laws, namely multi-symplectic conservation law

$$\partial_t(d\psi\wedge du)+\partial_x(d\psi\wedge dw+du\wedge dp+dv\wedge dq)=0, \tag{2.3}$$

local energy conservation law (LECL)

$$\partial_t\Big(\frac{1}{2}\alpha v^2+F+\frac{1}{2\beta}(E^2-q^2)\Big)+\partial_x(w\psi_t+pu_t+qv_t)=0, \tag{2.4}$$

and local momentum conservation law

$$\partial_t(u^2)+\partial_x\Big(\frac{1}{2}\alpha v^2+\frac{1}{2\beta}q^2+wu+pv+F(u,v)+\frac{1}{2\beta}(2q+E(u,v))E(u,v)-u\psi_t\Big)=0. \tag{2.5}$$

Under periodic boundary conditions, the above local conservation laws of the generalized fifth-order KdV equation can be integrated in x-direction to obtain the global symplectic, energy (GECL) and momentum conservation laws, i.e.,

$$\int\Big(d\psi\wedge du\Big)dx=C_1, \quad \int\Big(\frac{1}{2}\alpha v^2+F+\frac{1}{2\beta}(E^2-q^2)\Big)dx=C_2, \quad \int u^2dx=C_3,$$

where C_1, C_2 and C_3 are constants independent of t.

3 Properties of some difference operators

For the sake of simplicity, we introduce some notations: the spatial domain $\Lambda=[x_L,x_R]$ and $L=x_R-x_L$, $x_j=x_L+hj$, $t_n=n\tau$, $j=0,1,2,\cdots,N-1$, $n=0,1,2,\cdots$, where $h=L/N$, τ are spatial length and temporal step span. The approximation of the value of the function

$u(x,t)$ at the node (x_j,t_n) is denoted by u_j^n. In order to derive the algorithms conveniently, we also give some operators definition. Define the finite difference operators

$$\delta_t f_j^n = \frac{f_j^{n+1}-f_j^n}{\tau}, \quad \delta_x f_j^n = \frac{f_{j+1}^n-f_j^n}{h},$$

and averaging operators

$$A_t f_j^n = \frac{f_j^{n+1}+f_j^n}{2}, \quad A_x f_j^n = \frac{f_{j+1}^n+f_j^n}{2}.$$

The operators have the following properties:

• Commutative law

$$\delta_t\delta_x f_j^n = \delta_x\delta_t f_j^n, \quad A_tA_x f_j^n = A_xA_t f_j^n, \quad \delta_tA_x f_j^n = A_x\delta_t f_j^n, \quad A_t\delta_x f_j^n = \delta_xA_t f_j^n. \tag{3.1}$$

• Generalized discrete Leibnitz rule

$$\delta_x(f\cdot g)_j^n = \big(\epsilon f_{j+1}^n+(1-\epsilon)f_j^n\big)\cdot\delta_x g_j^n+\delta_x f_j^n\cdot\big((1-\epsilon)g_{j+1}^n+\epsilon g_j^n\big), \quad \forall 0\le\epsilon\le 1. \tag{3.2}$$

Some special cases for the generalized discrete Leibnitz rule are

$$\epsilon=0, \qquad \delta_x(f\cdot g)_j^n = f_j^n\cdot\delta_x g_j^n+\delta_x f_j^n\cdot g_{j+1}^n, \tag{3.3a}$$

$$\epsilon=\frac{1}{2}, \qquad \delta_x(f\cdot g)_j^n = A_x f_j^n\cdot\delta_x g_j^n+\delta_x f_j^n\cdot A_x g_j^n, \tag{3.3b}$$

$$\epsilon=1, \qquad \delta_x(f\cdot g)_j^n = f_{j+1}^n\cdot\delta_x g_j^n+\delta_x f_j^n\cdot g_j^n. \tag{3.3c}$$

Similarly, we can obtain a series of analogous discrete Leibniz's rules in the time direction. These Leibniz's rules play an important role in proving the local conservation law of the algorithms derived in the paper.

4 Energy-preserving algorithms for generalized fifth-order KdV equation

4.1 Local energy-preserving algorithm

In this section,we apply the AVF method in time and the implicit mid-point scheme in space to construct a local energy-preserving algorithm for generalized fifth-order KdV

equation. By some calculations, we can obtain a full-discrete scheme for Eq. (1.1)

$$\begin{cases} -\delta_t A_x u_j^n - \delta_x A_t w_j^n = 0, \\ \delta_t A_x \psi_j^n - \delta_x A_t p_j^n = A_x A_t w_j^n + F_1 + \dfrac{1}{\beta} E_1, \\ -\delta_x A_t q_j^n = \alpha A_x A_t v_j^n + A_x A_t p_j^n + F_2 + \dfrac{1}{\beta} E_2, \\ \delta_x A_t \psi_j^n = A_x A_t u_j^n, \\ \delta_x A_t u_j^n = A_x A_t v_j^n, \\ \delta_x A_t v_j^n = \dfrac{1}{\beta}(A_x A_t q_j^n + E_0), \end{cases} \tag{4.1}$$

where

$$\begin{aligned} &\widehat{\xi_u} \overset{\triangle}{=} (1-\xi) A_x u_j^n + \xi A_x u_j^{n+1}, \\ &F_1 = \int_0^1 F_u(\widehat{\xi_u}, \widehat{\xi_v}) d\xi, \quad F_2 = \int_0^1 F_v(\widehat{\xi_u}, \widehat{\xi_v}) d\xi, \quad E_0 = \int_0^1 E(\widehat{\xi_u}, \widehat{\xi_v}) d\xi, \\ &E_1 = \int_0^1 (\widehat{\xi_q} + E(\widehat{\xi_u}, \widehat{\xi_v})) E_u(\widehat{\xi_u}, \widehat{\xi_v}) d\xi, \quad E_2 = \int_0^1 (\widehat{\xi_q} + E(\widehat{\xi_u}, \widehat{\xi_v})) E_v(\widehat{\xi_u}, \widehat{\xi_v}) d\xi. \end{aligned}$$

The corresponding discrete conservation law for the scheme (4.1) can be defined as the LEP method. Now we analyze the local and global conservative properties of the LEP scheme.

Theorem 4.1. *The scheme* (4.1) *conserves the discrete local enenrgy conservation law*

$$\begin{aligned} &\delta_t \Big(\frac{1}{2} \alpha (A_x v_j^n)^2 + F(A_x u_j^n, A_x v_j^n) + \frac{1}{2\beta} (E^2(A_x u_j^n, A_x v_j^n) - (A_x q_j^n)^2) \Big) \\ &\quad + \delta_x (\delta_t \psi_j^n A_t w_j^n + \delta_t u_j^n A_t p_j^n + \delta_t v_j^n A_t q_j^n) = 0. \end{aligned} \tag{4.2}$$

Proof. Multiplying both sides of (4.1) by $\delta_t A_x \psi_j^n$, $\delta_t A_x u_j^n$, $\delta_t A_x v_j^n$, $\delta_t A_x w_j^n$, $\delta_t A_x p_j^n$, $\delta_t A_x q_j^n$, respectively, we have

$$\begin{cases} -\delta_t A_x \psi_j^n \delta_t A_x u_j^n - \delta_x A_t w_j^n \delta_t A_x \psi_j^n = 0, \\ \delta_t A_x u_j^n \delta_t A_x \psi_j^n - \delta_t A_x u_j^n \delta_x A_t p_j^n = \delta_t A_x u_j^n A_x A_t w_j^n + F_1 \delta_t A_x u_j^n + \dfrac{1}{\beta} E_1 \delta_t A_x u_j^n, \\ -\delta_t A_x v_j^n \delta_x A_t q_j^n = \alpha A_x A_t v_j^n \delta_t A_x v_j^n + A_x A_t p_j^n \delta_t A_x v_j^n + F_2 \delta_t A_x v_j^n + \dfrac{1}{\beta} E_2 \delta_t A_x v_j^n, \\ \delta_t A_x w_j^n \delta_x A_t \psi_j^n = A_x A_t u_j^n \delta_t A_x w_j^n, \\ \delta_t A_x p_j^n \delta_x A_t u_j^n = A_x A_t v_j^n \delta_t A_x p_j^n, \\ \delta_x A_t v_j^n \delta_t A_x q_j^n = \dfrac{1}{\beta} (A_x A_t q_j^n + E_0) \delta_t A_x q_j^n. \end{cases} \tag{4.3}$$

And we know the fact that

$$\begin{aligned}&F_1\delta_t A_x u_j^n + F_2\delta_t A_x v_j^n\\ =&\frac{1}{\tau}\int_0^1 \frac{d}{d\xi}F((1-\xi)A_x u_j^n+\xi A_x u_j^{n+1},(1-\xi)A_x v_j^n+\xi A_x v_j^{n+1})d\xi\\ =&\delta_t F(A_x u_j^n, A_x v_j^n).\end{aligned} \tag{4.4}$$

Adding each line of (4.3), the left-hand side reads

$$\begin{aligned}&\delta_t(\delta_x\psi_j^n A_x w_j^n)-\delta_x(\delta_t\psi_j^n A_t w_j^n)+\delta_t(\delta_x u_j^n A_x p_j^n)\\ &\quad-\delta_x(\delta_t u_j^n A_t p_j^n)+\delta_t(\delta_x v_j^n A_x q_j^n)-\delta_x(A_t q_j^n \delta_t v_j^n),\end{aligned} \tag{4.5}$$

and the right-hand side yields

$$\begin{aligned}&\delta_t F(A_x u_j^n, A_x v_j^n)+\frac{\alpha}{2}\delta_t(A_x v_j^n)^2+\frac{1}{2\beta}\delta_t(E^2(A_x u_j^n, A_x v_j^n)\\ &\quad-(A_x q_j^n)^2)+\delta_t(A_x p_j^n A_x v_j^n)+\delta_t(A_x w_j^n A_x u_j^n).\end{aligned} \tag{4.6}$$

(4.5) and (4.6) imply the discrete local energy conservation law (4.2). □

Remark 4.1. In the above derivation, we use the Commutative law (3.1) and the Leibnitz rule (3.2). The discrete LECL is consistent with the continuous LECL and independent of the boundary conditions.

Corollary 4.1. *Under the periodic boundary conditions, the LEP scheme preserves the discrete global energy conservation law*

$$\varepsilon^{n+1}=\varepsilon^n=\cdots\varepsilon^1=\varepsilon^0, \tag{4.7}$$

where

$$\varepsilon^n = h\sum_{j=0}^{N-1}\Big(\frac{1}{2}\alpha(A_x v_j^n)^2+F(A_x u_j^n, A_x v_j^n)+\frac{1}{2\beta}(E^2(A_x u_j^n, A_x v_j^n)-(A_x q_j^n)^2)\Big).$$

To prove it, we just need to sum (4.2) over all space index j and utilize the periodic boundary conditions.

4.2 Global energy-preserving algorithm

The main idea of the Fourier pseudospectral method is to approximate the partial differential operators ∂_x with the Fourier spectral differential matrix D_1 can replace the first-order differential operator ∂_x. Here, D_1 is an $N\times N$ skew-symmetric with elements

$$(D_1)_{j,k}=\begin{cases}\dfrac{1}{2}\mu(-1)^{j+k}\cot\Big(\mu\dfrac{x_j-x_k}{2}\Big), & j\neq k,\\ 0, & j=k,\end{cases}$$

where $j,k=1,2,\cdots,N$ and $\mu=2\pi/L$. Note that D_1 is an antisymmetric matrix, which can be calculated by using the discrete Fourier transform. For more details, see [27] and references therein.

Applying the Fourier pseudospectral method to space derivatives and the AVF method to time detivatives in the Eq. (1.1) gives

$$\begin{cases} -\delta_t u_j^n-(D_1A_t\mathbf{w}^n)_j=0, \\ \delta_t\psi_j^n-(D_1A_t\mathbf{p}^n)_j=A_tw_j^n+\widetilde{F_1}+\dfrac{1}{\beta}\widetilde{E_1}, \\ -(D_1A_t\mathbf{q}^n)_j=\alpha A_tv_j^n+A_tp_j^n+\widetilde{F_2}+\dfrac{1}{\beta}\widetilde{E_2}, \\ (D_1A_t\boldsymbol{\psi}^n)_j=A_tu_j^n, \\ (D_1A_t\mathbf{u}^n)_j=A_tv_j^n, \\ (D_1A_t\mathbf{v}^n)_j=\dfrac{1}{\beta}(A_tq_j^n+\widetilde{E_0}), \end{cases} \tag{4.8}$$

where

$$\begin{aligned} &\widetilde{\xi_u}\overset{\triangle}{=}(1-\xi)u_j^n+\xi u_j^{n+1}, \\ &\widetilde{F_1}=\int_0^1 F_u(\widetilde{\xi_u},\widetilde{\xi_v})d\xi,\quad \widetilde{F_2}=\int_0^1 F_v(\widetilde{\xi_u},\widetilde{\xi_v})d\xi,\quad \widetilde{E_0}=\int_0^1 E(\widetilde{\xi_u},\widetilde{\xi_v})d\xi, \\ &\widetilde{E_1}=\int_0^1(\widetilde{\xi_q}+E(\widetilde{\xi_u},\widetilde{\xi_v}))E_u(\widetilde{\xi_u},\widetilde{\xi_v})d\xi,\quad \widetilde{E_2}=\int_0^1(\widetilde{\xi_q}+E(\widetilde{\xi_u},\widetilde{\xi_v}))E_v(\widetilde{\xi_u},\widetilde{\xi_v})d\xi. \end{aligned}$$

Theorem 4.2. *With periodic boundary condition, the scheme* (4.8) *satisfies the discrete global energy conservation law exactly, namely*

$$\varepsilon^{n+1}=\varepsilon^n=\cdots\varepsilon^1=\varepsilon^0, \tag{4.9}$$

where

$$\varepsilon^n=h\sum_{j=0}^{N-1}\Big(\frac{\alpha}{2}((v_j^n)^2+F(u_j^n,v_j^n)+\frac{1}{2\beta}(E^2(u_j^n,v_j^n)-(q_j^n)^2)\Big).$$

Proof. We note the fact that

$$\begin{aligned} \widetilde{F_1}\delta_t u_j^n+\widetilde{F_2}\delta_t v_j^n&=\frac{1}{\tau}\int_0^1\frac{d}{d\xi}F((1-\xi)u_j^n+\xi u_j^{n+1},(1-\xi)v_j^n+\xi v_j^{n+1})d\xi \\ &=\delta_t F(u_j^n,v_j^n), \end{aligned}$$

and

$$\frac{1}{\beta}\widetilde{E_1}\delta_t u_j^n+\frac{1}{\beta}\widetilde{E_2}\delta_t v_j^n=\frac{1}{\beta}\delta_t(E(u_j^n,v_j^n)q_j^n)-\frac{1}{\beta}\widetilde{E_0}\delta_t q_j^n+\frac{1}{2\beta}\delta_t E^2(u_j^n,v_j^n).$$

Taking product with $\delta_t\psi_j^n$, $\delta_t u_j^n$, $\delta_t v_j^n$, $\delta_t w_j^n$, $\delta_t p_j^n$, $\delta_t q_j^n$ on both sides of Eq. (4.8) and then summing up each line, the left-hand side gives

$$-(D_1A_t\mathbf{w}^n)_j\delta_t\psi_j^n-(D_1A_t\mathbf{p}^n)_j\delta_t u_j^n-(D_1A_t\mathbf{q}^n)_j\delta_t v_j^n-A_tw_j^n(D_1\delta_t\boldsymbol{\psi}^n)_j-A_tp_j^n(D_1\delta_t\mathbf{u}^n)_j$$
$$-A_tq_j^n(D_1\delta_t\mathbf{v}^n)_j+\delta_t((D_1\boldsymbol{\psi}^n)_jw_j^n+(D_1\mathbf{v}^n)_jq_j^n+(D_1\mathbf{u}^n)_jp_j^n),$$

and the right-hand side reads

$$\delta_tF(u_j^n,v_j^n)+\frac{\alpha}{2}\delta_t(v_j^n)^2+\frac{1}{2\beta}\delta_t(E^2(u_j^n,v_j^n)-(q_j^n)^2).$$

By the properties of difference operator, calculating the left-hand side gives

$$-\sum_{k=0}^{N-1}(D_1)_{j,k}A_{k,j},$$

where

$$A_{k,j}=A_tw_k^n\delta_t\psi_j^n+A_tp_k^n\delta_tu_j^n+A_tq_k^n\delta_tv_j^n+A_tw_j^n\delta_t\psi_k^n+A_tp_j^n\delta_tu_k^n+A_tq_j^n\delta_tv_k^n.$$

Note that D_1 is antisymmetric and $A_{j,k}=A_{k,j}$, we can deduce

$$\sum_{j=0}^{N-1}\delta_t\Big(F(u_j^n,v_j^n)+\frac{\alpha}{2}(v_j^n)^2+\frac{1}{2\beta}(E^2(u_j^n,v_j^n)-(q_j^n)^2)\Big)=0,$$

which implies the discrete global energy conservation law (4.9). □

Remark 4.2. The method is called as the GEP method. According to the process of the above proof, we find that the differential matrix D_1 does not satisfy the discrete Leibnitz rule, therefore the method does not conserve the discrete local energy conservation law.

5 Linear stability analysis

To study the stability of the numerical method, that is, the sensitivity of the numerical solution to perturbations in the initial data, the von Neumann stability analysis is used. But the method is applicable only for linear problems. To apply this method, we assume that

$$u_j^n=\lambda^n e^{ijkh} \tag{5.1}$$

is the test function, where $i^2=-1$, λ is the amplification factor and k is wave number. The necessary condition for stability of the difference system is $|\lambda|\leq 1$. To apply von Neumann stability analysis properly, we consider the linearized the fifth-order KdV equation. The fifth-order KdV equation can be written as a linear system

$$Mz_t+Kz_x=Az,\quad z\in\mathbb{R}^6,\quad (x,t)\in\mathbb{R}^2, \tag{5.2}$$

where A is symmetric and

$$z=\begin{pmatrix}\psi\\ u\\ v\\ w\\ p\\ q\end{pmatrix},\quad A=\begin{pmatrix}0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&\alpha&0&1&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&\dfrac{1}{\beta}\end{pmatrix}.$$

The linearized version of the proposed scheme LEP applied to (5.2)

$$\begin{cases}-\delta_t A_x u_j^n-\delta_x A_t w_j^n=0,\\ \delta_t A_x \psi_j^n-\delta_x A_t p_j^n=A_x A_t w_j^n,\\ 0-\delta_x A_t q_j^n=\alpha A_x A_t v_j^n+A_x A_t p_j^n,\\ 0+\delta_x A_t \psi_j^n=A_x A_t u_j^n,\\ 0+\delta_x A_t u_j^n=A_x A_t v_j^n,\\ 0+\delta_x A_t v_j^n=\dfrac{1}{\beta}A_x A_t q_j^n.\end{cases}\tag{5.3}$$

Eliminating the auxiliary variables ψ, v, w, p, q yield an equivalent scheme

$$2\delta_t A_x^5 u_j^n+\alpha\delta_x^3 A_t A_x^2 u_j^n+\beta\delta_x^5 A_t u_j^n=0.\tag{5.4}$$

By substituting (5.1) into scheme (5.4), we get after some manipulation

$$\lambda=\frac{m}{n},\tag{5.5}$$

where

$$\begin{aligned}&r=\frac{2\alpha\tau}{h^3},\quad s=\frac{8\beta\tau}{h^5},\\ &m=(1+r+s)e^{i5kh}+(5-r-5s)e^{i4kh}+(10-2r+10s)e^{i3kh}\\ &\qquad+(10+2r-10s)e^{i2kh}+(5+r+5s)e^{ikh}+1-r-s,\\ &n=(1-r-s)e^{i5kh}+(5+r+5s)e^{i4kh}+(10+2r-10s)e^{i3kh}\\ &\qquad+(10-2r+10s)e^{i2kh}+(5-r-5s)e^{ikh}+1+r+s.\end{aligned}$$

By the some simple calculations, we can obtain that $|\lambda|=1$. This means that scheme (5.4) is unconditionally stable in the linear sense according to von Neumann stability analysis.

6 Numerical experiments

In this section, we present some numerical results for the Kawahara and the modified Kawahara equations.

6.1 Solitary waves

Firstly, we consider first numerical approximations of solitary wave solutions for the Kawahara equation and modified Kawahara equation. More precisely, we consider the Kawahara equation

$$u_t+uu_x+u_{xxx}-u_{xxxxx}=0,\quad u(x,0)=u_{ex}(x,0), \tag{6.1}$$

where

$$u_{ex}(x,t)=\frac{105}{169}\text{sech}^4\Big(\frac{1}{2\sqrt{13}}\Big(x-\frac{36t}{169}-x_0\Big)\Big) \tag{6.2}$$

is an exact solution solution of (6.1), and the modified Kawahara equation equation

$$u_t+u_x+u^2u_x+pu_{xxx}+qu_{xxxxx}=0,\quad u(x,0)=u_{ex}(x,0), \tag{6.3}$$

where

$$u_{ex}(x,t)=\pm\frac{3p}{\sqrt{-10q}}\text{sech}^2\Big(\frac{1}{2}\sqrt{\frac{-p}{5q}}\Big(x-\frac{25q-4p^2}{25q}t-x_0\Big)\Big) \tag{6.4}$$

is an exact solution solution of (6.3) and p, q are two parameters.

In order to apply the energy-preserving method, we fix $x_0=0$ and restrict the problem to the finite interval $[-L,L]$ with L sufficiently large such that the solution $u_{ex}(\pm L,t)$, $\partial_x u_{ex}(\pm L,t)$, $\partial_x^2 u_{ex}(L,t)$ are essentially zero for $t\in[0,T]$ (where T is given).

We apply the scaling $\widetilde{x}=L^{-1}x$, $\widetilde{t}=L^{-1}t$, and for the sake of simplicity, still use (x,t) to denote $(\widetilde{x},\widetilde{t})$. Then we consider the following scaled Kawahara equation.

$$u_t+uu_x+\frac{1}{L^2}u_{xxx}-\frac{1}{L^4}u_{xxxxx}=0,\quad x\in(-1,1), \tag{6.5a}$$

$$u(\pm1)=u_x(\pm1)=u_{xx}(1)=0, \tag{6.5b}$$

$$u(x,0)=\frac{105}{169}\text{sech}^4\Big(\frac{L}{2\sqrt{13}}x\Big), \tag{6.5c}$$

and the modified Kawahara equation

$$u_t+u_x+u^2u_x+\frac{1}{L^2}u_{xxx}-\frac{1}{L^4}u_{xxxxx}=0,\quad x\in(-1,1), \tag{6.6a}$$

$$u(\pm1)=u_x(\pm1)=u_{xx}(1)=0, \tag{6.6b}$$

$$u(x,0)=\frac{3}{\sqrt{10}}\text{sech}^2\Big(\frac{L}{2\sqrt{5}}x\Big). \tag{6.6c}$$

6.1.1 The Kawahara equation

For Eq. (6.5), we fix the parameters $L=200$, $\alpha=2/L^2$, $\beta=-2/L^4$ and the function $E=0$, $F=-u^3/3$. We choose $T=1$ and $\tau=10^{-4}$. Now, we show the energy preservation of the proposed scheme. Eqs. (6.5) and (6.6) with periodic boundary condition have global energy conservation law

$$\int \varepsilon(x,t)dx=C_2, \quad \varepsilon=\frac{\alpha}{2}u_x^2-\frac{\beta}{2}u_{xx}^2+F,$$

where C_2 is constant which is independent of t.

Define the errors in discrete global energy as

$$GE=h\sum_{j=0}^{N-1}(\varepsilon_j^n-\varepsilon_j^0),$$

where

$$\varepsilon_j^n=\frac{\alpha}{2}(D_1U_n)_j^2-\frac{\beta}{2}(D_1^2U^n)_j^2+F_j^n.$$

In order to apply our proposed method, we choose $T=1$ and $\tau=$1.0e-4. The convergence order is calculated by the following formula

$$\text{Order}=\frac{\ln(\text{error}_1/\text{error}_2)}{\ln(\text{h}_1/\text{h}_2)},$$

where h_i, error_i denote step size and the corresponding discrete error respectively. To test the time accuracy, we fix the spatial length with $N=1024$. The errors of u and numerical order of accuracy are listed in Table 1. Table 1 clearly indicates that the proposed method is solved efficiently and of second-order in time. The error due to spatial discretization decreases as the number of grid points N increases. The spectral accuracy of the method in space is demonstrated in Table 2.

In order to test the proposed method, we solve Eq. (6.5) for $t\in(0,10)$ (which corresponds to the real time $t\in(0,2000)$). In Fig. 1(b), the computation is done on the space interval $[-64,64]$ with grid number $N=256$ and time step $\tau=0.0001$. Figs. 1(b)

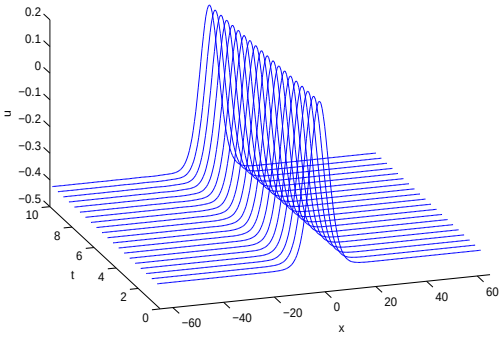
Table 1: The error due to temporal discretization decreases as time step τ is decreased ($T=1$, $N=1024$).

τ	L^2-error	Order	L^∞-error	Order
0.001	1.3444e-005	—	6.3695e-005	—
$0.001\cdot2^{-1}$	3.3610e-006	2.0000	1.5932e-005	1.9992
$0.001\cdot2^{-2}$	8.4025e-007	2.0000	3.9826e-006	2.0002
$0.001\cdot2^{-3}$	2.1006e-007	2.0000	9.9569e-007	1.9999
$0.001\cdot2^{-4}$	5.2516e-008	2.0000	2.4893e-007	2.0000
$0.001\cdot2^{-5}$	1.3129e-008	2.0000	6.2232e-008	2.0000

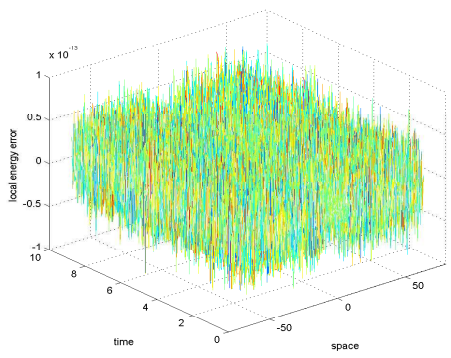
Table 2: The error due to spatial discretization decreases quickly as the number of grid points N increases ($T=1$, $\tau=$1.0e-4).

N	L^2-error	Order	L^∞-error	Order
140	1.5392e-003	—	2.3963e-003	—
160	2.1662e-004	14.6847	3.0662e-004	15.3979
180	1.9500e-005	20.4422	3.4073e-005	18.6537
200	3.7233e-006	15.7156	7.0249e-006	14.9870

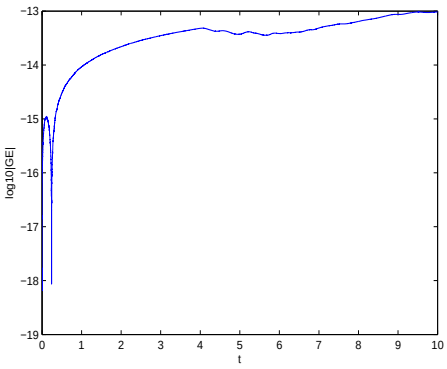
and (c) show that the local energy and the global energy is conserved with LEP method. Figs. 2(a) and (b) display the changes of errors of discrete global energy for GEP method with $N=256$ and $N=1024$. Fig. 3 shows the L^2 error and L^∞ error of the solution with $N=256$ ($N=1024$) and $\tau=$1.0e-4 by using the GEP method. It can be seen from the result that, we know that these proposed methods conserve the energy conservation law.



(a)

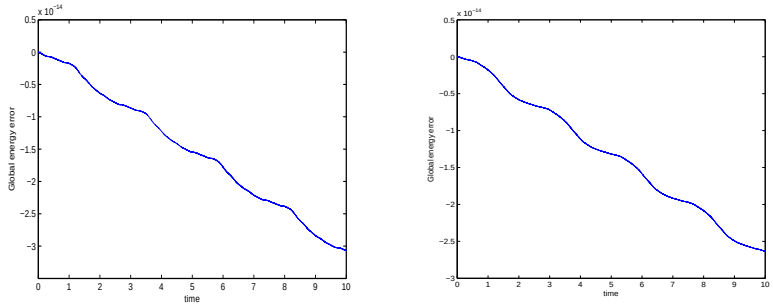


(b)



(c)

Figure 1: (a) The soliton solution of generalized fifth-order KdV equation. (b) Changes of errors of discrete local energy with time $0\leq t\leq 10$, the LEP method with $N=256$ and $\tau=0.0001$. (c) Changes of errors of discrete global energy with time $0\leq t\leq 10$, the GEP method with $N=256$ and $\tau=0.0001$.



(a) (b)

Figure 2: (a) The global error of discrete total energy of the GEP method with $N=256$ and $\tau=0.0001$; (b) The global error of discrete global energy of the GEP method with $N=1024$ and $\tau=0.0001$.

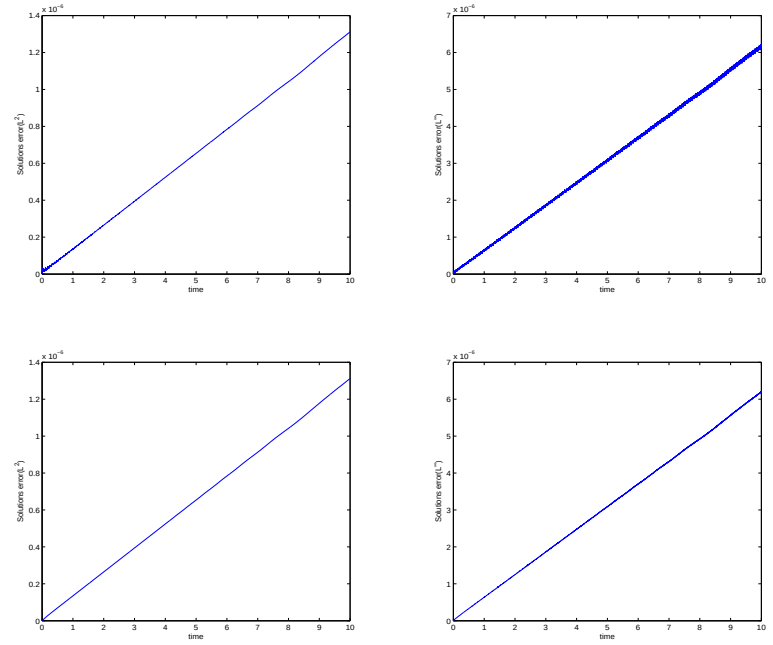
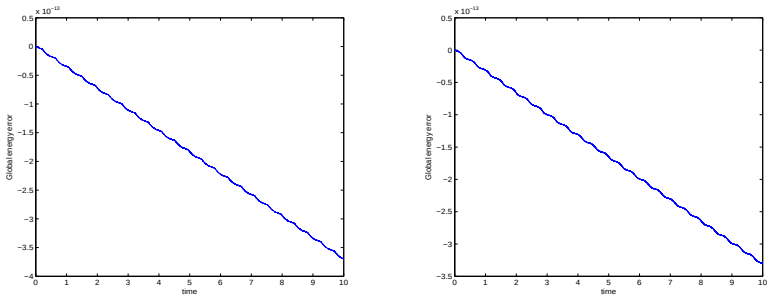


Figure 3: For the Kawahara equation, maximum solution errors L^∞ (right) and average solution errors L^2 (left) with $N=256$ (upper) and $N=1024$ (lower) ($\tau=$1.0e-4).

6.1.2 The modified Kawahara equation

For Eq. (6.6), we fix the parameters $L=200$, $\alpha=2/L^2$, $\beta=-2/L^4$ and the function $E=0$, $F=-u^2-1/6u^4$. Fig. 4 shows the variation of the solution errors obtained with $N=256$ ($N=1024$) and $\tau=$1.0e-4 by using the GEP method.



(a) (b)

Figure 4: (a) The global error of discrete total energy of the GEP method for the Modified Kawahara equation with $N=256$ and $\tau=0.0001$; (b) The global error of discrete total energy of the GEP method for the Modified Kawahara equation with $N=1024$ and $\tau=0.0001$.

6.2 Oscillatory solitary waves

We now consider the following Kawahara equation [29]

$$u_t-6uu_x-u_{xxx}-u_{xxxxx}=0, \tag{6.7}$$

which has the following asyptotic solution

$$\begin{aligned} u_{ex}(x,t)=&\sqrt{\frac{2}{19}}\epsilon\cos\theta\,\mathrm{sech}\,X+\epsilon^2\Big\{\frac{187}{57\sqrt{19}}\sin\theta\,\mathrm{sech}X\tanh X\\ &-\frac{4}{19}\Big(3+\frac{1}{3}\cos2\theta\Big)\mathrm{sech}^2X\Big\}+\mathcal{O}(\epsilon^3)\\ =&\overline{u}(x,t)+\mathcal{O}(\epsilon^3), \end{aligned} \tag{6.8}$$

where $\theta=\sqrt{0.5}(x-0.25t)$, $X=\epsilon(x-0.25t)$, $0<\epsilon\ll1$.

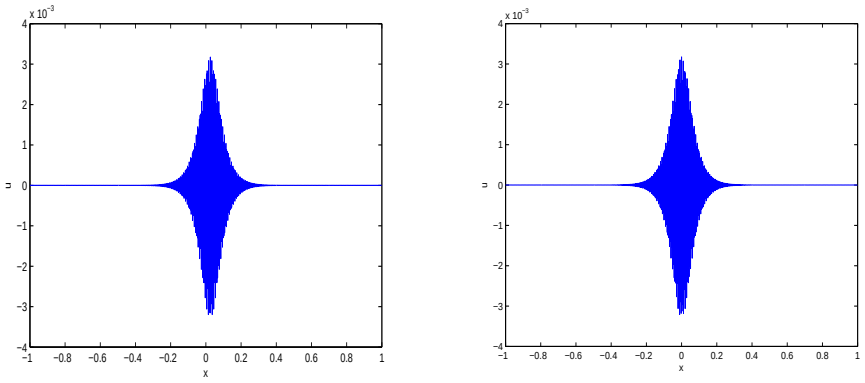
We rescale (6.7) with $(\widetilde{x},\widetilde{t})=(L^{-1}x,L^{-1}t)$ and still use (x,t) to denote $(\widetilde{x},\widetilde{t})$, we can obtain the following initial-boundary value problem:

$$u_t-6uu_x-\frac{1}{L^2}u_{xxx}-\frac{1}{L^4}u_{xxxxx}=0, \tag{6.9a}$$

$$u(\pm1,t)=u_x(\pm,t)=u_{xx}(1,t)=0, \tag{6.9b}$$

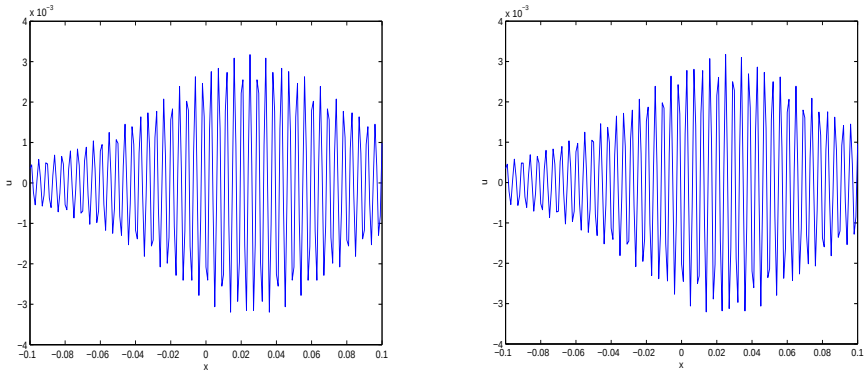
$$u(x,0)=\overline{u}(Lx,0). \tag{6.9c}$$

In our numerical experiments, we set $\epsilon=0.01$, $L=2000$, $\alpha=-2/L^2$, $\beta=-2/L^4$ and the function $E=0$, $F=2u^3$. Note that for smaller ϵ, larger L is needed to ensure that the boundary conditions (6.9b) are sufficiently accurate. In all the computations presented below, we use $\tau=$1.0e-5 and $N=2000$. The asymptotic solution at $t=0.1$ which corresponds to original time $t=200$. Note that the accuracy is limited by the accuracy of the asymptotic solution which is accurate to the order of ϵ^3.



(a) (b)

Figure 5: (a) The asymptotic solution of the oscillatory Kawahara equation and (b) the numerical solution with $\epsilon=0.01$ and $t=0.1$.



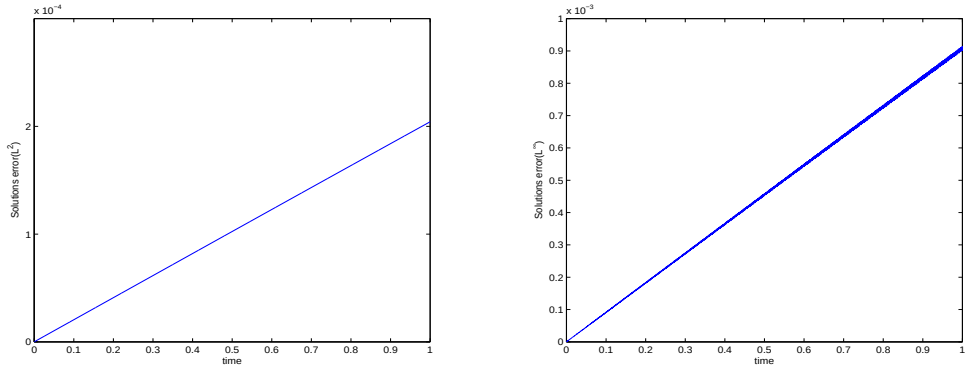
(a) (b)

Figure 6: (a) The asymptotic solution of the oscillatory Kawahara equation and (b) the numerical solution with $\epsilon=0.01$ and $t=0.1$.

In Figs. 5-6, we plot the numerical solutions and the asymptotic solutions at $t=0.1$ on the whole interval and a shorter interval. Figs. 7(a) and (b) show the solution errors obtained with $N=2000$ ($N=1024$) and $\tau=$1.0e-5. Fig. 8 shows the error in discrete global energy. We can see that the method preserves the global energy very well.

7 Conclusions

Based on the multi-symplectic formulation, we proposed two new energy-preserving methods for the generalized fifth-order KdV equation, i.e., a local energy-preserving algorithm and a global energy-preserving algorithm. The mid-point and the AVF method are used to construct the local energy-preserving algorithm which is independent of the



(a) (b)

Figure 7: (a) L^{∞} error of the asymptotic solution for the oscillatory Kawahara equation; (b) L^2 error of the asymptotic solution for the oscillatory Kawahara equation.

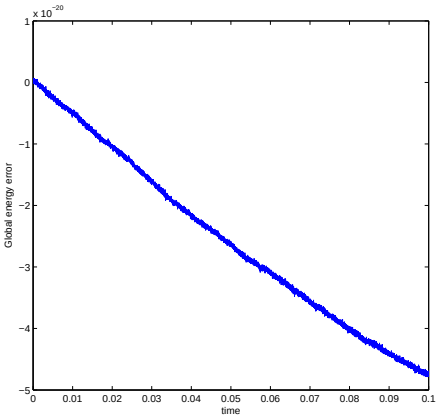


Figure 8: The global energy error with $N=2000$ and τ=1.0e-5.

boundary conditions. We note that the key of the global energy-preserving algorithm is to discretize first order differential operator ∂_x with Fourier pseudospectral differential matrix and apply the AVF method in time. The energy conservation properties of the new methods are proved and the von Neumann stability is also analysed. Numerical results show that the methods we proposed for the generalized fifth-order KdV equation are very effective.

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