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Two New Energy-Preserving Algorithms for Generalized Fifth-Order KdV Equation

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Abstract. In this paper, based on the multi-symplectic formulations of the generalized fifth-order KdV equation and the averaged vector field method, two new energy-preserving methods are proposed, including a new local energy-preserving algorithm which is independent of the boundary conditions and a new global energy-preserving method. We prove that the proposed methods preserve the energy conservation laws exactly. Numerical experiments are carried out, which demonstrate that the numerical methods proposed in the paper preserve energy well.

AMS subject classifications: 65M10, 78A48

Key words: Generalized fifth-order KdV equation, local energy-preserving, global energy-preserving, average vector field, Fourier pseudospectral.

1 Introduction

The generalized fifth-order KdV equation, also known as generalized Kawahara equation, is a typical model equation for phasma waves, capillary-gravity water waves, as well as other dispersive phenomena when the dispersion in the cubic KdV-type equation is weak. The equation can be written in the general form

$$2u_t+\alpha u_{xxx}+\beta u_{xxxxx}=\partial_x f(u,u_x,u_{xx}),\quad (x,t)\in[-L,L]\times(0,T], \tag{1.1}$$

where α and β are real parameters with $\beta\neq 0$ and $f(u,u_x,u_{xx})$ is a smooth function. The fifth-order term is called the Kawahara term, and its coefficient β is known as the dispersion coefficient. Due to Kawahara term, it is very difficult to compute the solutions

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of these equations accurately and efficiently. The difficulty, associated with the fifth-order term, is due to fact that for $\alpha<0$ solutions of these equations exhibit highly oscillatory behaviors. This equation was first presented by Kawahara in 1972. Much attention has been paid to the Kwahara-type equations in the last few years. Various numerical methods have been developed for this type of equations, including the sinc method and the decomposition series method [22], the Dual-Petrov-Galerkin method [29], the multi-symplectic Preissmann method [21], the semi-explicit finite differnece scheme [7], the meshless method [1] and so on.

The conservation of energy is a crucial property for the generalized fifth-order KdV equation and Eq. (1.1) has the following energy conservation law

$$\int \Big(\frac{1}{2}\alpha v^2+F+\frac{1}{2\beta}(E^2-q^2)\Big)dx,$$

where

$$f(u,v,s)=F_u(u,v)-vF_{uv}(u,v)-sF_{vv}(u,v)+2sE_u(u,v)+svE_{uv}(u,v)+v^2E_{uu}(u,v).$$

Therefore, it is natural to require a discretization to reflect this property. However, there has been few approaches which can preserve the energy of the original equation in the literature. This motivates our study to introduce energy-preserving methods for solving generalized fifth-order KdV equation. Energy-preserving method is a geometric method that can preserve one or more physical/geometric properties of the system exactly. Feng [14, 15] first presented the concept of symplectic schemes for Hamiltonian systems and further the structure-preserving algorithms for the general conservative dynamical systems. Afterwards, Marsden et al. [23], Bridges [2] and Reich [3] introduced the concept of multi-symplectic integrators based on a multi-symplectic structure of some conservative PDEs. In some fields, it is more convenient to construct numerical algorithms that preserve the energy conservation law rather than the symplectic or multi-symplectic ones [20].

Nowadays, energy-preserving methods have been successfully applied to various aspects on numerical PDEs [4,10–13,16,26]. Furihata [17] presented the discrete variational derivative methods for a large class of PDEs that inherit energy conservation or dissipation property. Matsuo and Furihata [24] generalized the discrete variational derivative methods for complex-valued nonlinear PDEs. Recently, Celledoni et al. [8] used the averaged vector field (AVF) method to construct a class of systematic energy-preserving methods based on symplectic formulation of Hamiltonian PDEs.

The AVF method appeared firstly in [25]. It is identified as energy-preserving and as a B-series method in [9]. For ordinary differential equation

$$\dot{y}=f(y),\quad y\in\mathbb{R}^d, \tag{1.2}$$

the AVF method is the map $y\longmapsto y'$ defined by

$$\frac{y'-y}{\tau}=\int_0^1 f((1-\xi)y+\xi y')d\xi, \tag{1.3}$$