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Asymptotic Study of a Boundary Value Problem Governed by the Elasticity Operator with Nonlinear Term

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Abstract. In this paper, a nonlinear boundary value problem in a three dimensional thin domain with Tresca's friction law is considered. The small change of variable $z = x_3/\varepsilon$ transforms the initial problem posed in the domain Ω^ε into a new problem posed on a fixed domain Ω independent of the parameter ε. As a main result, we obtain some estimates independent of the small parameter. The passage to the limit on ε, permits to prove the results concerning the limit of the weak problem and its uniqueness.

AMS subject classifications: 35R35, 76F10, 78M35

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1 Introduction

In this work, we consider a nonlinear boundary value problem governed by partial differential equations which describe the evolution of linear elastic materials in a bounded domain $\Omega^\varepsilon \subset \mathbb{R}^3$ with Tresca's friction law over a portion of the border and Dirichlet boundary conditions on the top and the lateral parts. However, this time we consider a nonlinear term $|u^\varepsilon|^\rho u^\varepsilon$, $\rho = p-2$ for $p > 1$. Thus we shall give the analogue of [3], where

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the authors gave the existence and the uniqueness of a non-Newtonian and incompressible fluid with stress tensor $\sigma_{ij}^{\varepsilon} = -p\delta_{ij} + 2\mu|D(u^{\varepsilon})|^{r-2}d_{ij}(u^{\varepsilon})$, in a thin domain and the extension of [5, 9]. Before stating the scientific context and our results, we first introduce some notations used in the paper. The boundary Γ^{ε} of the domain is assumed to be Lipschitz continuous so that the unit outward normal n exists almost everywhere on Γ^{ε}. The boundary of the domain is composed of three portions: ω the bottom of the domain, Γ_1^{ε} the upper surface, and Γ_L^{ε} the lateral surface. Similar studies have been made by several authors but with the usual boundary conditions, we cite for example: In [7], J. L. Lions studied theoretically a problem governed by the Laplace equation with Dirichlet boundary conditions. He proved the existence of a solution based essentially on the method of compactness, and the uniqueness of the solution by imposing conditions on the data. In [9], the authors, studied the similar nonlinear hyperbolic boundary value problem governed by partial differential equations which describe the evolution of the linear elastic materials but with Dirichlet-Neumann usual boundary conditions. They used the techniques of [7, 8] for a particular problem by replacing the elasticity equation by the Laplace operator and with the Neumann boundary conditions. The study of the asymptotic analysis of the same problem but in the particular case where $\rho = 1$ has been considered in [5]. In the last few years, some research papers have been written dealing with the asymptotic analysis of an incompressible fluid in a three-dimensional thin domain, when one dimension of the fluid domain tends to zero, (see e.g., [1, 3, 4]) and the references cited therein. More recently, the authors in [2] have studied the asymptotic analysis of a dynamical problem of isothermal elasticity with non linear friction of Tresca type but without the intervention of the nonlinear term. In [10] they studied the asymptotic behaviour of a dynamical problem of non-isothermal elasticity materials. The paper is structured as follows. In Section 2 we present some notations and give the problem statement and variational formulation. In Section 3 we use the asymptotic analysis, in which the small parameter is the height of the domain. We establish some estimates, independent on the parameter ε. These estimates will be useful in order to prove the convergence of the displacement toward the expected function. In Section 4, we investigate the convergence results of the limit weak problem and its uniqueness.

2 Problem statement and variational formulation

Let ω be a fixed bounded domain of $\mathbb{R}^3$ of equation $x_3 = 0$. We suppose that ω has a Lipschitz continuous boundary and is the bottom of the domain. The upper surface $\overline{\Gamma}_1^{\varepsilon}$ is defined by $x_3 = \varepsilon h(x) = \varepsilon h(x_1, x_2)$. We introduce a small parameter ε, that will tend to zero, and a function h on the closure of ω such that $0 < h_{\min} \le h(x) \le h_{\max}$, for all $(x,0)$ in ω. We study the asymptotic behaviour of an elasticity in the domain:

$$\Omega^{\varepsilon} = \{(x, x_3) \in \mathbb{R}^3 : (x,0) \in \omega,\ 0 < x_3 < \varepsilon h(x)\},$$

and Γ^{ε} its boundary : $\Gamma^{\varepsilon} = \overline{\Gamma}_1^{\varepsilon} \cup \overline{\Gamma}_L^{\varepsilon} \cup \overline{\omega}$, where $\overline{\Gamma}_L^{\varepsilon}$ is the lateral boundary.

Thus, the classical formulation of the mechanical problem is written as follows:
Problem P1 Find a displacement field $u^\varepsilon:\Omega\rightarrow\mathbb{R}^3$ such that

$$\frac{\partial\sigma^\varepsilon_{ij}}{\partial x_j}+|u^\varepsilon|^\rho u^\varepsilon+f^\varepsilon_i=0,\quad \text{where } \rho=p-2,\ p>1 \qquad \text{in } \Omega^\varepsilon, \tag{2.1a}$$

$$\sigma^\varepsilon_{ij}(u^\varepsilon)=2\mu d_{ij}(u^\varepsilon)+\lambda d_{kk}(u^\varepsilon)\delta_{ij} \qquad \text{in } \Omega^\varepsilon, \tag{2.1b}$$

$$u^\varepsilon=0 \qquad \text{on } \Gamma^\varepsilon_1, \tag{2.1c}$$

$$u^\varepsilon=g,\quad \text{with } g_3=0 \qquad \text{on } \Gamma^\varepsilon_L, \tag{2.1d}$$

$$u^\varepsilon\cdot n=0 \qquad \text{on } \omega, \tag{2.1e}$$

$$\left.\begin{array}{l}|\sigma^\varepsilon_\tau|<k^\varepsilon\Longrightarrow u^\varepsilon_\tau=s,\\ |\sigma^\varepsilon_\tau|=k^\varepsilon\Longrightarrow \exists\beta\geq 0,\text{ such that } u^\varepsilon_\tau=s-\beta\sigma^\varepsilon_\tau,\end{array}\right\} \qquad \text{on } \omega, \tag{2.1f}$$

where

$$f^\varepsilon=(f^\varepsilon_i)_{1\leq i\leq 3},\quad g=(g_i)_{1\leq i\leq 3},\quad \sigma^\varepsilon,\quad k^\varepsilon,\quad \delta_{ij}\quad \text{and}\quad d(u^\varepsilon)=\frac{1}{2}\Big(\frac{\partial u^\varepsilon_i}{\partial x_j}+\frac{\partial u^\varepsilon_j}{\partial x_i}\Big),$$

respectively, the body forces, the vector function, such that $\int_{\Gamma^\varepsilon}g\cdot n d\sigma=0$, the stress field, friction coefficient, the symbol of Kronecker and the symmetric deformation velocity tensor. Furthermore, the Eqs. (2.1a) and (2.1b) represent respectively, the equilibrium equation, the elastic behavior law. The formulae (2.1c) and (2.1d) are the displacement boundary conditions, in which $n=(n_1,n_2,n_3)$ denotes the unit outward normal vector on Γ^ε. Finally, condition (2.1e) represents the contact with Tresca's friction law given by Eq. (2.1f).

Now, to proceed with the variational formulation, we need the following functions spaces:

$$(W^{1,p}(\Omega^\varepsilon))^3=\Big\{v\in(L^p(\Omega^\varepsilon))^3:\frac{\partial v^\varepsilon_i}{\partial x_j}\in L^p(\Omega^\varepsilon),\ \text{for } 1<p<\infty \text{ and } i,j=1,2,3\Big\}.$$

$W^{1,p}_0(\Omega^\varepsilon)$ is the closure of $D(\Omega^\varepsilon)$ in $W^{1,p}(\Omega^\varepsilon)$. The dual space of $W^{1,p}_0(\Omega^\varepsilon)$ is $W^{-1,q}(\Omega^\varepsilon)$, where $p^{-1}+q^{-1}=1$. Let

$$W^{1,p}_{\Gamma^\varepsilon_1\cup\Gamma^\varepsilon_L}(\Omega^\varepsilon)=\big\{\psi\in W^{1,p}(\Omega^\varepsilon):\psi=0\ \text{ on }\ \Gamma^\varepsilon_1\cup\Gamma^\varepsilon_L\big\}.$$

Due to (2.1c) it is well known that there exists a function G^ε (see [1]) such that

$$G^\varepsilon\in(W^{1,p}(\Omega^\varepsilon))^3,\quad G^\varepsilon=g\quad \text{on } \Gamma^\varepsilon.$$

To get a weak formulation, we introduce the closed convex set

$$K^\varepsilon=\big\{v\in(W^{1,p}(\Omega^\varepsilon))^3:v=0\ \text{ on }\ \Gamma^\varepsilon_1,\ v=g\ \text{ on }\ \Gamma^\varepsilon_L\ \text{ and }\ v\cdot n=0\ \text{ on }\ \omega\big\}.$$

For every element $u^\varepsilon\in(W^{1,p}(\Omega^\varepsilon))^3$ we denote by u_n^ε and u_τ^ε the normal and the tangential components of u^ε on the boundary ω given by:

$$u_n^\varepsilon=u^\varepsilon\cdot n,\quad u_{\tau_i}^\varepsilon=u_i^\varepsilon-u_n^\varepsilon\cdot n_i.$$

Also, for a regular function σ^ε, we define its normal and tangential components by

$$\sigma_n^\varepsilon=(\sigma^\varepsilon\cdot n_i)\cdot n_j,\quad \sigma_{\tau_i}^\varepsilon=\sigma_{ij}^\varepsilon\cdot n_j-(\sigma_n^\varepsilon)\cdot n_i.$$

By standard calculations, the variational formulation of the Problem P1 is given by:

Problem P2 Find a displacement field $u^\varepsilon\in K^\varepsilon$ such that

$$a(u^\varepsilon,\varphi-u^\varepsilon)+B(u^\varepsilon,u^\varepsilon,\varphi-u^\varepsilon)+J^\varepsilon(\varphi)-J^\varepsilon(u^\varepsilon)\geq(f^\varepsilon,\varphi-u^\varepsilon),\quad\forall\varphi\in K^\varepsilon,\tag{2.2}$$

where

$$\begin{aligned}
&a(u,v)=2\mu\int_{\Omega^\varepsilon}d_{ij}(u)d_{ij}(v)dx'+\lambda\int_{\Omega^\varepsilon}div(u)div(v)dx',\\
&J^\varepsilon(v)=\int_\omega k^\varepsilon|v-s|dx,\quad\forall v\in H^1(\Omega^\varepsilon)^3,\\
&B(u^\varepsilon,v)=\int_{\Omega^\varepsilon}|u^\varepsilon|^\rho u^\varepsilon v dx',\\
&(f,v)=\int_{\Omega^\varepsilon}f_iv_idx',\quad\forall v\in H^1(\Omega^\varepsilon)^3.
\end{aligned}$$

We also denote by the nonlinear operator

$$(A(u^\varepsilon),\phi)=a(u^\varepsilon,\phi)+B(u^\varepsilon,\phi).\tag{2.3}$$

Lemma 2.1. *Problems P1 and P2 are equivalent.*

Proof. The proof is similar to [5] for $\rho=1$. □

The existence and uniqueness results of the weak solution to the Problem P2 is obtained in the following theorem.

Theorem 2.1. *Assuming that $f\in L^q(\Omega^\varepsilon)^3$, k^ε is a positive function in $L^\infty(\omega)$ and $h\in L^\infty(\omega)\cap C^1(\omega)$, there exists a unique solution $u^\varepsilon\in K^\varepsilon$ to Problem P2.*

Proof. The Problem P2 is still written

$$\left\{\begin{array}{l}
\text{Find } u^\varepsilon\in K^\varepsilon \text{ such that}\\
a(u^\varepsilon,\phi-u^\varepsilon)+B(u^\varepsilon,\phi-u^\varepsilon)+j^\varepsilon(\phi)-j^\varepsilon(u^\varepsilon)+\delta_{K^\varepsilon}(\phi)\\
\quad-\delta_{K^\varepsilon}(u^\varepsilon)\geq(f^\varepsilon,\phi-u^\varepsilon),\quad\forall\phi\in(W^{1,p}(\Omega^\varepsilon))^3,
\end{array}\right.$$

where δ_{K^ε} is the function given by

$$\delta_{K^\varepsilon}=\left\{\begin{array}{ll}0, & \text{if } v\in K^\varepsilon,\\ +\infty, & \text{if } v\notin K^\varepsilon.\end{array}\right.$$

The proof is based on the nonlinear operators theory (see for example [7]): It is enough to show that the operator A given by (2.3) is bounded coercive semicontinuous and strictly monotone, and that $J^\varepsilon+\delta_{K^\varepsilon}$ is a convex and proper functional. □

3 Transposing of the Problem P1

We shall now focus our attention on the asymptotic analysis for the Problem P1. For this, we transform this problem into an equivalent problem on a domain Ω independent of the parameter ε via the rescaling $z=x_3/\varepsilon$ (as in [1]). So, for (x,x_3) in Ω^ε, we have (x,z) in

$$\Omega=\big\{(x,z)\in\mathbb{R}^3,\ (x,0)\in\omega \ \text{ and } \ 0<z<h(x)\big\},$$

and we denote by $\Gamma=\bar{\omega}\cup\Gamma_L\cup\Gamma_1$ its boundary, then we define the following functions in Ω

$$\hat{u}_3^\varepsilon(x,z)=\varepsilon^{-1}u_3^\varepsilon(x,x_3) \quad \text{and} \quad \hat{u}_i^\varepsilon(x,z)=u_i^\varepsilon(x,x_3), \quad i=1,2, \tag{3.1a}$$

$$\hat{k}=\varepsilon k^\varepsilon, \quad \hat{f}(x,z)=\varepsilon^2 f^\varepsilon(x,x_3) \quad \text{and} \ \hat{g}(x,z)=g^\varepsilon(x,x_3), \tag{3.1b}$$

$$\hat{G}_i(x,z)=G_i^\varepsilon(x,x_3), \quad i=1,2, \quad \text{and} \quad \hat{G}_3(x,z)=\varepsilon^{-1}G_3^\varepsilon(x,x_3). \tag{3.1c}$$

Let

$$\begin{aligned}
&K=\big\{\varphi\in(W^{1,p}(\Omega))^3:\varphi=(\varphi_1,\varphi_2,\varphi_3),\ \varphi=\hat{G} \ \text{ on } \ \Gamma_1\cup\Gamma_L,\ \varphi\cdot n=0 \ \text{ on } \ \omega\big\},\\
&\Pi(K)=\big\{\bar{\varphi}\in(W^{1,p}(\Omega))^2:\bar{\varphi}=(\varphi_1,\varphi_2),\ \varphi_i=\hat{G}_i \ \text{ on } \ \Gamma_1\cup\Gamma_L,\ i=1,2,\big\},\\
&V_z=\Big\{v=(v_1,v_2)\in(L^p(\Omega))^2:\frac{\partial v_i}{\partial z}\in L^p(\Omega),\ i=1,2, \ \text{ on } \ \Gamma_1\Big\},
\end{aligned}$$

where V_z is the Banach space with norm

$$|v|_{V_z}=\Big(\sum_{i=1}^2\Big(|v_i|^2+\Big|\frac{\partial v_i}{\partial z}\Big|^2\Big)\Big)^{\frac{1}{2}}.$$

Assuming (3.1), then problem (2.2) leads to the following form:

$$\begin{cases}
\text{Find } \hat{u}^\varepsilon\in K, \text{ such that}\\
a(\hat{u}^\varepsilon,\hat{\varphi}-\hat{u}^\varepsilon)+B(\hat{u}^\varepsilon,\varphi-\hat{u}^\varepsilon)+J(\hat{\varphi})-J(\hat{u}^\varepsilon)\\
\quad \geq \sum\limits_{i=1}^2(\hat{f}_i,\hat{\varphi}_i-\hat{u}_i^\varepsilon)+\varepsilon(\hat{f}_3,\hat{\varphi}_3-\hat{u}_3^\varepsilon), \quad \forall\hat{\varphi}\in K,
\end{cases} \tag{3.2}$$

where

$$\begin{aligned}
&J(\hat{\varphi})=\int_\omega \hat{k}|\hat{\varphi}-s|dx,\\
&B(\hat{u}^\varepsilon,\varphi-\hat{u}^\varepsilon)=\varepsilon^2\sum_{i=1}^2\int_\Omega|\hat{u}_i^\varepsilon|^\rho\hat{u}_i^\varepsilon(\hat{\phi}_i-\hat{u}_i^\varepsilon)dx'dz+\varepsilon^{p+1}\int_\Omega|\hat{u}_3^\varepsilon|^\rho\hat{u}_3^\varepsilon(\hat{\phi}_3-\hat{u}_3^\varepsilon)dx'dz,\\
&a(\hat{u}^\varepsilon,\hat{\varphi}-\hat{u}^\varepsilon)=\mu\varepsilon^2\sum_{i,j=1}^2\int_\Omega\Big(\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\varepsilon}{\partial x_i}\Big)\frac{\partial}{\partial x_j}(\hat{\varphi}_i-\hat{u}_i^\varepsilon)dxdz\\
&\qquad\qquad\qquad +\mu\sum_{i=1}^2\int_\Omega\Big(\frac{\partial\hat{u}_i^\varepsilon}{\partial z}+\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\Big)\frac{\partial}{\partial z}(\hat{\varphi}_i-\hat{u}_i^\varepsilon)dxdz
\end{aligned}$$

$$
\begin{aligned}
&+\mu\varepsilon^2\sum_{i=1}^{2}\int_\Omega\Big(\frac{\partial\hat{u}_i^\varepsilon}{\partial z}+\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(\hat{\varphi}_3-\hat{u}_3^\varepsilon)dxdz\\
&+2\mu\varepsilon^2\int_\Omega\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\cdot\frac{\partial}{\partial z}(\hat{\varphi}_3-\hat{u}_3^\varepsilon)dxdz\\
&+\lambda\varepsilon^2\int_\Omega div(\hat{u}^\varepsilon)\cdot div(\hat{\varphi}-\hat{u}^\varepsilon)dxdz.
\end{aligned}
$$

In the next, we will obtain first estimates on $\hat{u}^\varepsilon$. These estimates will be useful in order to prove the convergence of $\hat{u}^\varepsilon$ toward the expected function.

Theorem 3.1. *Under the same assumptions as in Theorem 2.1, there exists a constant C independent of ε such that*

$$
\sum_{1\le i,j\le 2}\Big|\varepsilon\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}\Big|^p_{L^p(\Omega)}+\Big|\varepsilon\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\Big|^p_{L^p(\Omega)}+\sum_{i=1}^{2}\Big(\Big|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}\Big|^p_{L^p(\Omega)}+\Big|\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\Big|^p_{L^p(\Omega)}\Big)\le C. \tag{3.3}
$$

Proof. Let u^ε be a solution to the problem (2.2). We deduce

$$
a(u^\varepsilon,u^\varepsilon)\le a(u^\varepsilon,\varphi)-B(u^\varepsilon,u^\varepsilon)+B(u^\varepsilon,\varphi)+J^\varepsilon(\varphi)-J^\varepsilon(u^\varepsilon)+(f^\varepsilon,u^\varepsilon)-(f^\varepsilon,\varphi),\quad \forall\varphi\in K^\varepsilon.
$$

As $B(u^\varepsilon,u^\varepsilon)>0$ and $J^\varepsilon(u^\varepsilon)$ is positive, (since $k>0$), we have

$$
a(u^\varepsilon,u^\varepsilon)\le a(u^\varepsilon,\varphi)+B(u^\varepsilon,\varphi)+J^\varepsilon(\varphi)+(f^\varepsilon,u^\varepsilon)-(f^\varepsilon,\varphi),\quad \forall\varphi\in K^\varepsilon.
$$

From Korn's inequality, there exists a constant $C_K>0$ independent of ε, such that

$$
a(u^\varepsilon,u^\varepsilon)\ge 2\mu C_K|\nabla u^\varepsilon|^p_{L^p(\Omega)}. \tag{3.4}
$$

By the Hölder inequality and the Young inequality, we obtain

$$
\begin{aligned}
a(u^\varepsilon,\varphi)\le&\int_{\Omega^\varepsilon}2\mu|d_{ij}(u^\varepsilon)||d_{ij}(\varphi)|dxdx_3+\lambda\int_{\Omega^\varepsilon}|div(u^\varepsilon)||div(\varphi)|dxdx_3\\
\le&\Big(\int_{\Omega^\varepsilon}\Big(\frac{\mu pC_K}{2}\Big)^{\frac{1}{p}}|d_{ij}(u^\varepsilon)|\Big)\Big(\int_{\Omega^\varepsilon}\frac{4}{(pC_K)^{\frac{1}{p}}}\Big(\frac{\mu}{2}\Big)^{\frac{1}{q}}|d_{ij}(\varphi)|dxdx_3\Big)\\
&+\Big(\int_{\Omega^\varepsilon}\Big(\frac{\mu pC_K}{4}\Big)^{\frac{1}{p}}|div(u^\varepsilon)|\Big)\Big(\int_{\Omega^\varepsilon}\lambda\Big(\frac{\mu pC_K}{4}\Big)^{-\frac{1}{p}}|div(\varphi)|dxdx_3\Big)\\
\le&\frac{\mu C_K}{2}|d_{ij}(u^\varepsilon)|^p_{L^p(\Omega^\varepsilon)}+\frac{2^{2q-1}\mu}{q(pC_K)^{\frac{q}{p}}}|d_{ij}(\varphi)|^q_{L^q(\Omega^\varepsilon)}\\
&+\frac{\mu C_K}{4}\int_{\Omega^\varepsilon}|div(u^\varepsilon)|^pdxdx_3+\frac{\lambda^q}{q(\frac{\mu pC_K}{4})^{\frac{q}{p}}}\int_{\Omega^\varepsilon}|div(\varphi)|^qdxdx_3\\
\le&\frac{\mu C_K}{2}|\nabla u^\varepsilon|^p_{L^p(\Omega^\varepsilon)}+\frac{2^{2q-1}\mu}{q(pC_K)^{\frac{q}{p}}}|\nabla\varphi|^q_{L^q(\Omega^\varepsilon)}\\
&+\frac{\mu C_K}{4}\int_{\Omega^\varepsilon}|\nabla u^\varepsilon|^pdxdx_3+\frac{\lambda^q}{q(\frac{\mu pC_K}{4})^{\frac{q}{p}}}\int_{\Omega^\varepsilon}|\nabla\varphi|^qdxdx_3.
\end{aligned}
$$

Then

$$|a(u^\varepsilon,\varphi)|\leq\frac{3}{4}\mu C_K|\nabla u^\varepsilon|^p_{L^p(\Omega^\varepsilon)}+\Big(\frac{2^{2q-1}\mu}{q(pC_K)^{\frac{q}{p}}}+\frac{\lambda^q}{q(\frac{\mu pC_K}{4})^{\frac{q}{p}}}\Big)|\nabla\varphi|^q_{L^q(\Omega^\varepsilon)}, \tag{3.5}$$

by the Poincaré inequality, we obtain

$$|u^\varepsilon|_{L^p(\Omega^\varepsilon)}\leq\varepsilon h_{\max}|\nabla u^\varepsilon|_{L^p(\Omega^\varepsilon)}. \tag{3.6}$$

We have by the Young inequality and (3.6):

$$\begin{aligned}|f^\varepsilon|_{L^q(\Omega^\varepsilon)}|u^\varepsilon|_{L^p(\Omega^\varepsilon)}&\leq(\varepsilon h_{\max}(\mu pC_K2)^{-\frac{1}{p}}|f^\varepsilon|_{L^q(\Omega^\varepsilon)})\Big(\Big(\frac{\mu pC_K}{2}\Big)^{\frac{1}{p}}|\nabla u^\varepsilon|_{L^p(\Omega^\varepsilon)}\Big)\\&\leq\frac{\mu C_K}{2}|\nabla u^\varepsilon|^p_{L^p(\Omega^\varepsilon)}+(\varepsilon h_{\max})^q q^{-1}\Big(\frac{\mu pC_K}{2}\Big)^{-\frac{q}{p}}|f^\varepsilon|^q_{L^q(\Omega^\varepsilon)}, &(3.7a)\\|f^\varepsilon|_{L^q(\Omega^\varepsilon)}|\varphi^\varepsilon|_{L^p(\Omega^\varepsilon)}&\leq\frac{\mu C_K}{2}|\nabla\varphi|^p_{L^p(\Omega^\varepsilon)}+(\varepsilon h_{\max})^q q^{-1}\Big(\frac{\mu pC_K}{2}\Big)^{-\frac{q}{p}}|f^\varepsilon|^q_{L^q(\Omega^\varepsilon)}. &(3.7b)\end{aligned}$$

On the other hand, by the Hölder, Young and Poincaré inequalities, we obtain

$$\begin{aligned}|B(u^\varepsilon,\phi)|&\leq\int_{\Omega^\varepsilon}\Big(\Big(\frac{\mu qC_k}{2\varepsilon h_{\max}}\Big)^{-\frac{1}{q}}|\phi|\Big)\Big(\Big(\frac{\mu qC_k}{2\varepsilon h_{\max}}\Big)^{\frac{1}{q}}|u^\varepsilon|^{(p-1)}\Big)dx\\&\leq\frac{1}{p(\frac{\mu qC_k}{2\varepsilon h_{\max}})^{\frac{p}{q}}}\int_{\Omega^\varepsilon}|\phi|^p dx+\frac{\mu C_k}{2\varepsilon h_{\max}}\int_{\Omega^\varepsilon}|u^\varepsilon|^p dx\\&\leq\frac{\mu C_k}{2}|\bigtriangledown u^\varepsilon|^p_{L^p(\Omega^\varepsilon)}+\frac{1}{p(\frac{\mu qC_k}{2\varepsilon h_{\max}})^{\frac{p}{q}}}|\bigtriangledown\phi|^p_{L^p(\Omega^\varepsilon)}.\end{aligned}$$

If $p\leq q$ then

$$|B(u^\varepsilon,\phi)|\leq\frac{\mu C_k}{2}|\bigtriangledown u^\varepsilon|^p_{L^p(\Omega^\varepsilon)}+\frac{h_{\max}^{\frac{p}{q}}}{p(\frac{\mu qC_k}{2})^{\frac{p}{q}}}|\bigtriangledown\phi|^p_{L^q(\Omega^\varepsilon)}. \tag{3.8}$$

Using (3.4)-(3.8) and choosing $\phi=G^\varepsilon$, we have

$$\begin{aligned}\frac{1}{4}\mu C_k|\nabla u^\varepsilon|^p_{L^p(\Omega^\varepsilon)}\leq&\Big(\frac{2^{2q-1}\mu}{q(pC_k)^{\frac{q}{p}}}+\frac{\lambda^q}{q(\frac{\mu pC_k}{4})^{\frac{q}{p}}}+\frac{h_{\max}^{\frac{p}{q}}}{p(\frac{\mu qC_k}{2})^{\frac{p}{q}}}+\frac{\mu C_k}{2}\Big)|\bigtriangledown G^\varepsilon|^q_{L^q(\Omega^\varepsilon)}\\&+\frac{2\varepsilon^q h^q_{\max}}{q(\frac{\mu pC_k}{2})^{\frac{q}{p}}}|f^\varepsilon|^q_{L^q(\Omega^\varepsilon)}. \end{aligned}\tag{3.9}$$

As

$$\varepsilon^q|f^\varepsilon|^q_{L^q(\Omega^\varepsilon)}=\varepsilon^{1-p}|\hat{f}^\varepsilon|^p_{L^p(\Omega^\varepsilon)}\quad\text{and}\quad\Big|\frac{\partial u_i^\varepsilon}{\partial x_3}\Big|^p_{L^p(\Omega^\varepsilon)}=\varepsilon^{1-p}\Big|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}\Big|^p_{L^p(\Omega^\varepsilon)},\quad i=1,2,$$

then we multiply (3.9) by ε^{p-1} we deduce (3.3), with

$$C=\frac{4}{\mu C_k}\Big[\Big(\frac{2^{2q-1}\mu}{q(pC_k)^{\frac{q}{p}}}+\frac{\lambda^q}{q(\frac{\mu pC_k}{4})^{\frac{q}{p}}}+\frac{h_{\max}^{\frac{p}{q}}}{p(\frac{\mu qC_k}{2})^{\frac{p}{q}}}+\frac{\mu C_k}{2}\Big)|\nabla\hat{G}|^q_{L^q(\Omega^\varepsilon)}\Big]$$
$$+\frac{4}{\mu C_k}\frac{2h_{\max}^q}{q(\frac{\mu pC_k}{2})^{\frac{q}{p}}}|\hat{f}|^q_{L^q(\Omega^\varepsilon)},$$

which completes the proof. □

4 The limit problem

Theorem 4.1. *Under the same assumptions of Theorem 3.1, there exists $\hat{u}_i^*\in V_z$, $i=1,2$, such that*

$$\hat{u}_i^\varepsilon \ harpoonup\ u_i^\star \ \ (1\le i\le 2) \qquad \textit{weakly in } V_z, \tag{4.1a}$$
$$\varepsilon\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}\ harpoonup\ 0 \ \ (1\le i,j\le 2) \qquad \textit{weakly in } L^p(\Omega), \tag{4.1b}$$
$$\varepsilon\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\ harpoonup\ 0 \qquad \textit{weakly in } L^p(\Omega), \tag{4.1c}$$
$$\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\ harpoonup\ 0 \ \ (1\le i\le 2) \qquad \textit{weakly in } L^p(\Omega). \tag{4.1d}$$

Proof. From the inequality (3.3) there exists a fixed constant C which does not depend on ε such that

$$\Big|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}\Big|^2_{L^2(\Omega)}\le C,\quad i=1,2.$$

Using the above estimate and the Poincaré inequality in the domain Ω, we deduce (4.1a). Also (4.1b)-(4.1d) follows from (3.3). □

Theorem 4.2. *With the same assumptions of Theorem 3.1, $\hat{u}^*$ satisfy*

$$\mu\sum_{i=1}^{2}\int_\Omega\frac{\partial\hat{u}_i^*}{\partial z}\cdot\frac{\partial}{\partial z}(\hat{\varphi}_i-\hat{u}_i^*)dxdz+J(\hat{\varphi})-J(\hat{u}_i^*)\ge\sum_{i=1}^{2}(\hat{f}_i,\hat{\varphi}_i-\hat{u}_i^*),\quad\forall\varphi\in\Pi(K), \tag{4.2a}$$
$$-\mu\frac{\partial^2\hat{u}_i^*}{\partial z^2}=\hat{f}_i,\quad \textit{for } i=1,2,\quad \textit{in } L^q(\Omega). \tag{4.2b}$$

Proof. The variational inequality (3.2) can be written as:

$$\sum_{i=1}^{4}I_i(\varepsilon)+\lambda\varepsilon^2\int_\Omega div(\hat{u}^\varepsilon)div(\hat{\phi}-\hat{u}^\varepsilon)dx'dz+\varepsilon^2\sum_{i=1}^{2}\int_\Omega|\hat{u}_i^\varepsilon|^\rho\hat{u}_i^\varepsilon(\hat{\phi}_i-\hat{u}_i^\varepsilon)dxdz$$
$$+\varepsilon^{p+1}\int_\Omega|\hat{u}_3^\varepsilon|^\rho\hat{u}_3^\varepsilon(\hat{\phi}_3-\hat{u}_3^\varepsilon)dxdz+\hat{\jmath}(\hat{\phi})-\hat{\jmath}(\hat{u}^\varepsilon)\ge\sum_{i=1}^{2}(\hat{f}_i,\hat{\phi}_i-\hat{u}_i^\varepsilon)+\varepsilon(\hat{f}_3,\hat{\phi}_3-\hat{u}_3^\varepsilon),$$

where

$$I_1=\mu\varepsilon^2\sum_{i,j=1}^{2}\Big(\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\varepsilon}{\partial x_i}\Big)\frac{\partial}{\partial x_j}(\hat{\varphi}_i-\hat{u}_i^\varepsilon)dxdz,$$

$$I_2=\mu\sum_{i=1}^{2}\int_\Omega\Big(\frac{\partial\hat{u}_i^\varepsilon}{\partial z}+\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\Big)\frac{\partial}{\partial z}(\hat{\varphi}_i+\hat{u}_i^\varepsilon)dxdz,$$

$$I_3=\mu\varepsilon^2\sum_{i=1}^{2}\int_\Omega\Big(\frac{\partial\hat{u}_i^\varepsilon}{\partial z}+\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(\hat{\varphi}_3-\hat{u}_3^\varepsilon)dxdz,$$

$$I_4=2\mu\varepsilon^2\int_\Omega\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\cdot\frac{\partial}{\partial z}(\hat{\varphi}_3-\hat{u}_3^\varepsilon)dxdz.$$

By the Theorem 4.1, we have

$$\lim_{\varepsilon\to0}\sum_{i=1}^{4}I_i(\varepsilon)=\mu\sum_{i=1}^{2}\int_\Omega\frac{\partial\hat{u}_i^*}{\partial z}\cdot\frac{\partial}{\partial z}(\hat{\varphi}_i-\hat{u}_i^*)dxdz,$$

$$\lim_{\varepsilon\to0}\int_\Omega\varepsilon\hat{f}_3\varphi dxdz=0,$$

$$\lim_{\varepsilon\to0}\varepsilon^2\sum_{i=1}^{2}\int_\Omega|\hat{u}_i^\varepsilon|^\rho\hat{u}_i^\varepsilon(\hat{\phi}_i-\hat{u}_i^\varepsilon)dxdz=0,$$

$$\lim_{\varepsilon\to0}\varepsilon^{p+1}\int_\Omega|\hat{u}_3^\varepsilon|^\rho\hat{u}_3^\varepsilon(\hat{\phi}_3-\hat{u}_3^\varepsilon)dxdz=0.$$

And as J is convex and lower semicontinuous i.e.,

$$\lim_{\varepsilon\to0}\Big(\inf\int_\omega\hat{k}|\hat{u}^\varepsilon-s|dx\Big)\geq\int_\omega\hat{k}|\hat{u}^*-s|dx,$$

we obtain

$$\mu\sum_{i=1}^{2}\int_\Omega\frac{\partial\hat{u}_i^*}{\partial z}\cdot\frac{\partial}{\partial z}(\hat{\varphi}_i-\hat{u}_i^*)dxdz+J(\hat{\varphi})-J(\hat{u}_i^*)\geq\sum_{i=1}^{2}(\hat{f}_i,\hat{\varphi}_i-\hat{u}_i^*). \tag{4.3}$$

We now choose in the variational inequality (4.3)

$$\hat{\phi}_i=u_i^\star\pm\psi_i,\quad \psi_i\in W_0^{1,p}(\Omega),\quad i=1,2,$$

to get

$$\mu\sum_{i=1}^{2}\int_\Omega\frac{\partial\hat{u}_i^*}{\partial z}\frac{\partial\psi_i}{\partial z}dxdz=\sum_{i=1}^{2}\int_\Omega\hat{f}_i\psi_idxdz.$$

Using now the Green formula, we deduce first with $\psi_1=0$ and $\psi_2\in W_0^{1,p}(\Omega)$, then $\psi_2=0$ and $\psi_1\in W_0^{1,p}(\Omega)$ the following equality:

$$-\int_\Omega\mu\frac{\partial}{\partial z}\Big(\frac{\partial\hat{u}_i^*}{\partial z}\Big)\psi_idxdz=\int_\Omega\hat{f}_i\psi_idxdz, \tag{4.4a}$$

$$-\mu\frac{\partial^2\hat{u}_i^*}{\partial z^2}=\hat{f}_i,\quad\text{for } i=1,2,\quad\text{in } W^{-1,q}(\Omega), \tag{4.4b}$$

and as $\hat{f}_i \in L^q(\Omega)$, then (4.4) is valid in $L^q(\Omega)$. □

Theorem 4.3. *Under the same hypothesis of Theorem 3.1 we have the following inequality*

$$\int_\omega k|\psi+s^*-s|-|s^*-s|dx-\int_\omega \mu\hat{\tau}^*\psi dx\geq 0, \quad \forall\psi\in(L^p(\omega))^2, \tag{4.5a}$$

$$\begin{cases} \mu|\hat{\tau}^*|<\hat{k}\rightarrow s^*=s, \\ \mu|\hat{\tau}^*|=\hat{k}\rightarrow \exists\beta>0, \text{ such that } \; s^*=s+\beta\hat{\tau}^*, \end{cases} \tag{4.5b}$$

where

$$\hat{\tau}^*=\frac{\partial\hat{u}^*}{\partial z}(x,0) \quad \text{and} \quad s^*(x)=\hat{u}^*(x,0).$$

Also the limit fonction $\hat{u}^*$ and s^* satisfy the following weak form of the Reynolds equation

$$\int_\omega \Big(\tilde{F}-\frac{h}{2}s^*+\int_0^h \hat{u}^*(x,z)dz\Big)\nabla\psi(x)dx=0, \quad \forall\phi\in W^{1,p}(\omega), \tag{4.6}$$

where

$$\tilde{F}(x)=\frac{1}{\mu}\int_0^h F(x,z)dz-\frac{h}{2\mu}F(x,h) \quad \text{and} \quad F(x,z)=\int_0^z\int_0^\zeta \hat{f}_i(x,\alpha)d\zeta d\alpha.$$

Proof. For the demonstration, it is enough to follow the same techniques of [1, 3] in the case of fluid. □

Theorem 4.4. *The solution $u^\star$ in V_z of inequalities* (4.2a)-(4.2b) *is unique.*

Proof. Let $\hat{u}^1$ and $\hat{u}^2$ be two solutions of (4.2a). Then

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial\hat{u}_i^1}{\partial z}\cdot\frac{\partial}{\partial z}(\varphi_i-\hat{u}_i^1)dxdz+J(\varphi)-J(\hat{u}_i^1)\geq\sum_{i=1}^2(\hat{f}_i,\varphi_i-\hat{u}_i^1), \tag{4.7a}$$

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial\hat{u}_i^2}{\partial z}\cdot\frac{\partial}{\partial z}(\varphi_i-\hat{u}_i^2)dxdz+J(\varphi)-J(\hat{u}_i^2)\geq\sum_{i=1}^2(\hat{f}_i,\varphi_i-\hat{u}_i^2). \tag{4.7b}$$

Taking $\varphi=\hat{u}^2$ in (4.7a) and $\varphi=\hat{u}^1$ in (4.7b), respectively, as test functions, we get

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial\hat{u}_i^1}{\partial z}\cdot\frac{\partial}{\partial z}(\hat{u}_i^2-\hat{u}_i^1)dxdz+J(\hat{u}^2)-J(\hat{u}_i^1)\geq\sum_{i=1}^2(\hat{f}_i,\hat{u}_i^2-\hat{u}_i^1), \tag{4.8a}$$

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial\hat{u}_i^2}{\partial z}\cdot\frac{\partial}{\partial z}(\hat{u}_i^1-\hat{u}_i^2)dxdz+J(\hat{u}^1)-J(\hat{u}_i^2)\geq\sum_{i=1}^2(\hat{f}_i,\hat{u}_i^1-\hat{u}_i^2). \tag{4.8b}$$

By summing the two inequalities (4.8a) and (4.8b), we obtain

$$\mu\sum_{i=1}^{2}\int_{\Omega}\frac{\partial\hat{u}_i^1}{\partial z}\cdot\frac{\partial}{\partial z}(\hat{u}_i^2-\hat{u}_i^1)dxdz+\mu\sum_{i=1}^{2}\int_{\Omega}\frac{\partial\hat{u}_i^2}{\partial z}\cdot\frac{\partial}{\partial z}(\hat{u}_i^1-\hat{u}_i^2)dxdz\geq 0,$$

$$\mu\sum_{i=1}^{2}\int_{\Omega}\frac{\partial}{\partial z}(\hat{u}_i^1-\hat{u}_i^2)\cdot\frac{\partial}{\partial z}(\hat{u}_i^1-\hat{u}_i^2)dxdz\leq 0,$$

this implies:

$$\mu\Big|\frac{\partial}{\partial z}(\hat{u}_i^1-\hat{u}_i^2)\Big|^2_{L^2(\Omega)}=0.$$

By the Poincaré inequality, we get

$$|\hat{u}_i^1-\hat{u}_i^2|_{V_z}\leq c\Big|\frac{\partial}{\partial z}(\hat{u}_i^1-\hat{u}_i^2)\Big|^2_{L^2(\Omega)}=0.$$

So

$$\hat{u}^1=\hat{u}^2.$$

This completed the proof. □

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