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A Second-Order Three-Level Difference Scheme for a Magneto-Thermo-Elasticity Model

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Abstract. This article deals with the numerical solution to the magneto-thermo-elasticity model, which is a system of the third order partial differential equations. By introducing a new function, the model is transformed into a system of the second order generalized hyperbolic equations. A priori estimate with the conservation for the problem is established. Then a three-level finite difference scheme is derived. The unique solvability, unconditional stability and second-order convergence in L_∞-norm of the difference scheme are proved. One numerical example is presented to demonstrate the accuracy and efficiency of the proposed method.

AMS subject classifications: 65M06, 65M12, 65M12, 78M20, 80M20

Key words: Magneto-thermo-elasticity, conservation, finite difference, solvability, stability, convergence.

1 Introduction

In the past decades, magneto-thermo-elastic theory has been widely applied in acoustics, geophysics, micro electromechanical systems (MEMS). There are some reviews about the classical and generalized theories [1–3]. The generalized thermo-elasticity theories were considered to be more realistic than the conventional theory in dealing with practical problems. Some models have been proposed in order to study the property of the analytical solution by the energy functional and generalized variational principle [4–8]. As is known to all, it is difficult to find the analytical solution for the generalized model. Thus, the numerical solutions are usually obtained by the numerical methods such as finite difference method [9–12], finite element method [13–16] and numerical integration method [17–19]. It should be mentioned that there are few papers concentrating on the

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analysis of the established numerical methods. This paper deals with the Green-Naghdi (G-N) model [20] derived by Green and Naghdi [21–23] who provided sufficient basic modifications in the constitutive equations. The model is written as follows:

$$R_M^2\frac{\partial^2 U}{\partial\xi^2}-\frac{\partial\theta}{\partial\xi}=\frac{\partial^2 U}{\partial\eta^2}, \tag{1.1a}$$

$$\frac{\partial^2\theta}{\partial\eta^2}+\varepsilon_T\frac{\partial^3\theta}{\partial\xi\partial\eta^2}=c_T^2\frac{\partial^2\theta}{\partial\xi^2}+\kappa_0\frac{\partial^3\theta}{\partial\xi^2\partial\eta}, \tag{1.1b}$$

where $U=U(\xi,\eta)$ and $\theta=\theta(\xi,\eta)$ are functions of displacement and temperature, ξ and η are space variable and time variable respectively. The constants R_M^2, ε_T, c_T and κ_0 are dimensionless quantities, where R_M^2 describes the impact of the external magnetic field in the process of thermo-elasticity, ε_T is the thermo-elasticity coupled coefficient, c_T is the wave velocity in G-N model, and κ_0 is the thermal diffusion coefficient. When $\kappa_0\ll c_T^2$, Eq. (1.1b) becomes

$$\frac{\partial^2\theta}{\partial\eta^2}+\varepsilon_T\frac{\partial^3\theta}{\partial\xi\partial\eta^2}=c_T^2\frac{\partial^2\theta}{\partial\xi^2}, \tag{1.2}$$

which corresponds to the thermo-elasticity undamped heat-wave solution in G-N model. For emphasizing the main idea, in this article, we consider the model with the simplified notations in the bounded domain. Then (1.1a) and (1.2) turn into

$$u_{tt}=au_{xx}-v_x,\qquad 0<x<1,\qquad 0<t\le T, \tag{1.3a}$$

$$v_{tt}=cv_{xx}-bu_{xtt},\qquad 0<x<1,\qquad 0<t\le T. \tag{1.3b}$$

Now taking the derivative with respect to t on both sides of the Eq. (1.3a), we have

$$u_{ttt}=au_{xxt}-v_{xt},\quad 0<x<1,\quad 0<t\le T.$$

Let $w=u_t$, then the above equation is equivalent to

$$w_{tt}=aw_{xx}-v_{xt},$$

and (1.3b) can be rewritten as

$$v_{tt}=cv_{xx}-bw_{xt}.$$

In the following we consider the numerical solution of initial boundary value problem for the coupled system:

$$w_{tt}=aw_{xx}-v_{xt}+g_1(x,t),\qquad 0<x<1,\quad 0<t\le T, \tag{1.4a}$$

$$v_{tt}=cv_{xx}-bw_{xt}+g_2(x,t),\qquad 0<x<1,\quad 0<t\le T, \tag{1.4b}$$

$$w(x,0)=\phi_1(x),\quad w_t(x,0)=\phi_2(x),\qquad 0\le x\le 1, \tag{1.4c}$$

$$v(x,0)=\psi_1(x),\quad v_t(x,0)=\psi_2(x),\qquad 0\le x\le 1, \tag{1.4d}$$

$$w(0,t)=\alpha_1(t),\quad w(1,t)=\alpha_2(t),\qquad 0<t\le T, \tag{1.4e}$$

$$v(0,t)=\beta_1(t),\quad v(1,t)=\beta_2(t),\qquad 0<t\le T, \tag{1.4f}$$

where $v=v(x,t)$ is the temperature, $w(x,t)$ is the derivative of displacement with respect to t, $\phi_1(0)=\alpha_1(0)$, $\phi_1(1)=\alpha_2(0)$, $\psi_1(0)=\beta_1(0)$, $\psi_1(1)=\beta_2(0)$, $\phi_2(0)=\alpha_1'(0)$, $\phi_2(1)=\alpha_2'(0)$, $\psi_2(0)=\beta_1'(0)$, $\psi_2(1)=\beta_2'(0)$, the positive constants $a=R_M^2$, $b=\varepsilon_T$, and $c=c_T^2$. We suppose that the problem (1.4a)-(1.4f) has a unique smooth solution.

This paper is organized as follows. The priori estimate of the solution of the problem (1.4a)-(1.4f) is set up in Section 2. A three level difference scheme is constructed and the unique solvability of the difference scheme is showed in Section 3. The unconditional stability and convergence of the difference scheme are proved in Section 4. In Section 5, the difference scheme is written into two kinds of matrix forms. One numerical example is presented in Section 6 to show the effectiveness of the presented difference scheme. The article ends with a short conclusion.

2 The priori estimate of the solutions of the differential equations

In this section we will provide a priori estimate of the problem (1.4a)-(1.4f).

Theorem 2.1. *Assume $w=w(x,t)$ and $v=v(x,t)$ are solutions of the following problem:*

$$
\begin{aligned}
&w_{tt}=aw_{xx}-v_{xt}+g_1(x,t), && 0<x<1,\quad 0<t\le T, && (2.1a)\\
&v_{tt}=cv_{xx}-bw_{xt}+g_2(x,t), && 0<x<1,\quad 0<t\le T, && (2.1b)\\
&w(x,0)=\phi_1(x),\quad w_t(x,0)=\phi_2(x), && 0\le x\le 1, && (2.1c)\\
&v(x,0)=\psi_1(x),\quad v_t(x,0)=\psi_2(x), && 0\le x\le 1, && (2.1d)\\
&w(0,t)=0,\qquad w(1,t)=0, && 0<t\le T, && (2.1e)\\
&v(0,t)=0,\qquad v(1,t)=0, && 0<t\le T, && (2.1f)
\end{aligned}
$$

where $\phi_1(0)=\phi_1(1)=\psi_1(0)=\psi_1(1)=\phi_2(0)=\phi_2(1)=\psi_2(0)=\psi_2(1)=0$ and a, b, c are positive constants. Then we have

$$
\begin{aligned}
&b\int_0^1 w_t^2(x,t)dx+\int_0^1 v_t^2(x,t)dx+ab\int_0^1 w_x^2(x,t)dx+c\int_0^1 v_x^2(x,t)dx\\
\le& e^t\Big\{b\int_0^1 \phi_2^2(x)dx+\int_0^1\psi_2^2(x)dx+ab\int_0^1[\phi_1'(x)]^2dx+c\int_0^1[\psi_1'(x)]^2dx\\
&+\int_0^t e^{-s}\Big[b\int_0^1 g_1^2(x,s)dx+\int_0^1 g_2^2(x,s)dx\Big]ds\Big\},\quad 0<t\le T. \qquad (2.2)
\end{aligned}
$$

If $g_1(x,t)\equiv 0$, $g_2(x,t)\equiv 0$, then the following conservation law holds:

$$
\begin{aligned}
&b\int_0^1 w_t^2(x,t)dx+\int_0^1 v_t^2(x,t)dx+ab\int_0^1 w_x^2(x,t)dx+c\int_0^1 v_x^2(x,t)dx\\
=&b\int_0^1 \phi_2^2(x)dx+\int_0^1\psi_2^2(x)dx+ab\int_0^1[\phi_1'(x)]^2dx+c\int_0^1[\psi_1'(x)]^2dx,\quad 0<t\le T. \qquad (2.3)
\end{aligned}
$$

Proof. Multiplying (2.1a) by bw_t, and integrating the result for x on $[0,1]$, it follows that

$$\begin{aligned}&\frac{b}{2}\frac{d}{dt}\Big(\int_0^1 w_t^2(x,t)dx+a\int_0^1 w_x^2(x,t)dx\Big)\\ =&-b\int w_t(x,t)v_{xt}(x,t)dx+b\int_0^1 w_t(x,t)g_1(x,t)dx,\quad 0<t\le T.\end{aligned}\tag{2.4}$$

Similarly, multiplying (2.1b) by v_t, and integrating the result for x from 0 to 1, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_0^1 v_t^2(x,t)dx+c\int_0^1 v_x^2(x,t)dx\Big)\\ =&-b\int v_t(x,t)w_{xt}(x,t)dx+\int_0^1 v_t(x,t)g_2(x,t)dx,\quad 0<t\le T.\end{aligned}\tag{2.5}$$

Denote

$$E(t)=b\int_0^1 w_t^2(x,t)dx+\int_0^1 v_t^2(x,t)dx+ab\int_0^1 w_x^2(x,t)dx+c\int_0^1 v_x^2(x,t)dx$$

and

$$G(t)=b\int_0^1 g_1^2(x,t)dx+\int_0^1 g_2^2(x,t)dx.$$

Adding (2.4) to (2.5) and using (2.1e)-(2.1f), it yields

$$\begin{aligned}\frac{dE(t)}{dt}=&-2b\int_0^1\big(w_t(x,t)v_{xt}(x,t)dx+v_t(x,t)w_{xt}(x,t)\big)dx\\ &+2b\int_0^1 w_t(x,t)g_1(x,t)dx+2\int_0^1 v_t(x,t)g_2(x,t)dx\\ =&-2bw_t(x,t)v_t(x,t)|_{x=0}^1+2b\int_0^1 w_t(x,t)g_1(x,t)dx+2\int_0^1 v_t(x,t)g_2(x,t)dx\\ =&2b\int_0^1 w_t(x,t)g_1(x,t)dx+2\int_0^1 v_t(x,t)g_2(x,t)dx,\quad 0<t\le T.\end{aligned}\tag{2.6}$$

Applying the Cauchy-Schwarz inequality to the above equality, we have

$$\frac{dE(t)}{dt}\le E(t)+G(t),\quad 0<t\le T.$$

Utilizing the Gronwall inequality, we obtain

$$E(t)\le e^t\Big[E(0)+\int_0^t e^{-s}G(s)ds\Big],\quad 0<t\le T.$$

Recalling the definition of $E(t)$, the estimate (2.2) holds.

When $g_1(x,t)\equiv 0$, $g_2(x,t)\equiv 0$, it follows from (2.6) that

$$\frac{dE(t)}{dt}=0,\quad 0\le t\le T.$$

Consequently,

$$E(t)=E(0), \quad 0\le t\le T.$$

This means that (2.3) is valid. The proof is completed. □

3 The derivation of the difference scheme and the unique solvability

For the finite difference approximation, we take two positive integers m and n, and let $h=1/m$ and $\tau=T/n$. The domain $[0,1]\times[0,T]$ is covered by the grid $\Omega_h\times\Omega_\tau$, where $\Omega_h=\{x_i|x_i=ih,\ 0\le i\le m\}$ and $\Omega_\tau=\{t_k|t_k=k\tau,\ 0\le k\le n\}$. In addition, denote $x_{i-1/2}=(x_i+x_{i-1})/2$ and $t_{k-1/2}=(t_k+t_{k-1})/2$. Suppose that $u=\{u_i^k|0\le i\le m,\ 0\le k\le n\}$ is a grid function defined on $\Omega_h\times\Omega_\tau$. Introduce the following notations:

$$\begin{aligned}
&u_{i+\frac12}^k=\frac12(u_i^k+u_{i+1}^k), \qquad \delta_x u_{i+\frac12}^k=\frac1h(u_{i+1}^k-u_i^k), \qquad \delta_t u_i^{k+\frac12}=\frac1\tau(u_i^{k+1}-u_i^k),\\
&\Delta_t u_i^k=\frac{1}{2\tau}(u_i^{k+1}-u_i^{k-1}), \qquad \Delta_x u_i^k=\frac{1}{2h}(u_{i+1}^k-u_{i-1}^k),\\
&\delta_x^2 u_i^k=\frac1h\Big(\delta_x u_{i+\frac12}^k-\delta_x u_{i-\frac12}^k\Big), \qquad \delta_t^2 u_i^k=\frac1\tau\Big(\delta_t u_i^{k+\frac12}-\delta_t u_i^{k-\frac12}\Big).
\end{aligned}$$

In addition, if $u_0^k=u_m^k=0$, we define

$$\|u^k\|_\infty=\max_{1\le i\le m-1}|u_i^k|, \quad \|u^k\|=\sqrt{h\sum_{i=1}^{m-1}(u_i^k)^2}, \quad |u^k|_1=\sqrt{h\sum_{i=0}^{m-1}(\delta_x u_{i+\frac12}^k)^2}.$$

Define the grid functions

$$U_i^k=u(x_i,t_k), \quad W_i^k=w(x_i,t_k), \quad V_i^k=v(x_i,t_k), \quad 0\le i\le m, \quad 0\le k\le n.$$

Utilizing the formula of Taylor expansion with integral remainder

$$g(\mu_0+\mu)=\sum_{l=0}^{k}\frac{g^{(l)}(\mu_0)}{l!}\mu^l+\frac{\mu^{k+1}}{k!}\int_0^1(1-\lambda)^k g^{(k+1)}(\mu_0+\lambda\mu)d\lambda, \quad g\in C^{k+1},$$

we have

$$\begin{aligned}
\frac{\partial u}{\partial x}(x_i,t_k)=\Delta_x U_i^k-\frac{h^2}{4}\int_0^1(1-\lambda)^2\Big[&\frac{\partial^3 u}{\partial x^3}(x_i+\lambda h,t_k)\\
&+\frac{\partial^3 u}{\partial x^3}(x_i-\lambda h,t_k)\Big]d\lambda, \quad 1\le i\le m-1, \quad 0\le k\le n,
\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial t}(x_i,t_k)=&\Delta_t U_i^k-\frac{\tau^2}{4}\int_0^1(1-s)^2\Big[\frac{\partial^3 u}{\partial t^3}(x_i,t_k+s\tau)\\&+\frac{\partial^3 u}{\partial t^3}(x_i,t_k-s\tau)\Big]ds,\quad 0\le i\le m,\quad 1\le k\le n-1,\\\frac{\partial u}{\partial t}(x_i,t_{\frac12})=&\delta_t U_i^{\frac12}-\frac{\tau^2}{16}\int_0^1(1-s)^2\Big[\frac{\partial^3 u}{\partial t^3}\Big(x_i,t_{\frac12}\\&+\frac{s\tau}{2}\Big)+\frac{\partial^3 u}{\partial t^3}\Big(x_i,t_{\frac12}-\frac{s\tau}{2}\Big)\Big]ds,\quad 0\le i\le m.\end{aligned}$$

Consequently, there exist ξ_1 and ξ_2 in $[x_{i-1},x_{i+1}]$ such that

$$\begin{aligned}u_{xt}(x_i,t_k)=&\Delta_x\Delta_t U_i^k-\frac{\tau^2}{4}\int_0^1(1-s)^2\Big[u_{xttt}(\xi_1,t_k+s\tau)ds+u_{xttt}(\xi_1,t_k-s\tau)\Big]ds\\&-\frac{h^2}{4}\int_0^1(1-\lambda)^2\Big[u_{xxxt}(x_i+\lambda h,t_k)+u_{xxxt}(x_i-\lambda h,t_k)\Big]d\lambda\\=&\Delta_x\Delta_t U_i^k+\mathcal{O}(\tau^2+h^2),\quad 1\le i\le m-1,\quad 1\le k\le n-1,\\u_{xt}(x_i,t_{\frac12})=&\Delta_x\delta_t U_i^{\frac12}-\frac{\tau^2}{16}\int_0^1(1-s)^2\Big[u_{xttt}\Big(\xi_2,t_{\frac12}+\frac{s\tau}{2}\Big)ds+u_{xttt}\Big(\xi_2,t_{\frac12}+\frac{s\tau}{2}\Big)\Big]ds\\&-\frac{h^2}{4}\int_0^1(1-\lambda)^2\Big[u_{xxxt}(x_i+\lambda h,t_{\frac12})+u_{xxxt}(x_i-\lambda h,t_{\frac12})\Big]d\lambda\\=&\Delta_x\delta_t U_i^{\frac12}+\mathcal{O}(\tau^2+h^2),\quad 1\le i\le m-1,\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^2 u}{\partial x^2}(x_i,t_k)=\delta_x^2 U_i^k+\mathcal{O}(h^2),&&1\le i\le m-1,&&0\le k\le n,\\&\frac{\partial^2 u}{\partial t^2}(x_i,t_k)=\delta_t^2 U_i^k+\mathcal{O}(\tau^2),&&0\le i\le m,&&1\le k\le n-1.\end{aligned}$$

Using the equalities above in (1.4a)-(1.4b) at the points (x_i,t_k) and $(x_i,t_{1/2})$, we obtain

$$\delta_t^2 W_i^k-\frac{a}{2}(\delta_x^2 W_i^{k+1}+\delta_x^2 W_i^{k-1})+\Delta_x\Delta_t V_i^k=g_1(x_i,t_k)+(R_1)_i^k,\quad 1\le i\le m-1,\quad 1\le k\le n-1,\tag{3.1a}$$

$$\delta_t^2 V_i^k-\frac{c}{2}(\delta_x^2 V_i^{k+1}+\delta_x^2 V_i^{k-1})+b\Delta_x\Delta_t W_i^k=g_2(x_i,t_k)+(R_2)_i^k,\quad 1\le i\le m-1,\quad 1\le k\le n-1,\tag{3.1b}$$

$$\frac{2}{\tau}\Big[\delta_t W_i^{\frac12}-\varphi_2(x_i)\Big]-a\delta_x^2 W_i^{\frac12}+\Delta_x\delta_t V_i^{\frac12}=g_1(x_i,t_0)+(R_1)_i^0,\quad 1\le i\le m-1,\tag{3.1c}$$

$$\frac{2}{\tau}\Big[\delta_t V_i^{\frac12}-\psi_2(x_i)\Big]-c\delta_x^2 V_i^{\frac12}+b\Delta_x\delta_t W_i^{\frac12}=g_2(x_i,t_0)+(R_2)_i^0,\quad 1\le i\le m-1.\tag{3.1d}$$

By the assumptions on the existence of smooth solutions, there exists a constant c_0 such that

$$|(R_l)_i^k|\le c_0(\tau^2+h^2),\qquad 1\le l\le 2,\qquad 1\le i\le m-1,\qquad 1\le k\le n-1,\tag{3.2a}$$

$$|(R_l)_i^0|\le c_0(\tau+h^2),\qquad 1\le l\le 2,\qquad 1\le i\le m-1.\tag{3.2b}$$

Noticing the initial and boundary conditions

$$W_i^0=\phi_1(x_i),\qquad V_i^0=\psi_1(x_i),\qquad 1\le i\le m-1, \tag{3.3a}$$

$$W_0^k=\alpha_1(t_k),\qquad W_m^k=\alpha_2(t_k),\qquad 0\le k\le n, \tag{3.3b}$$

$$V_0^k=\beta_1(t_k),\qquad V_m^k=\beta_2(t_k),\qquad 0\le k\le n, \tag{3.3c}$$

and omitting the small terms in (3.1a)-(3.1d), the difference scheme is constructed for the system (1.4a)-(1.4f) as follows:

$$\delta_t^2 w_i^k-\frac{a}{2}(\delta_x^2 w_i^{k+1}+\delta_x^2 w_i^{k-1})+\Delta_x\Delta_t v_i^k=g_1(x_i,t_k),\quad 1\le i\le m-1,\quad 1\le k\le n-1, \tag{3.4a}$$

$$\delta_t^2 v_i^k-\frac{c}{2}(\delta_x^2 v_i^{k+1}+\delta_x^2 v_i^{k-1})+b\Delta_x\Delta_t w_i^k=g_2(x_i,t_k),\quad 1\le i\le m-1,\quad 1\le k\le n-1, \tag{3.4b}$$

$$\frac{2}{\tau}\Big[\delta_t w_i^{\frac{1}{2}}-\phi_2(x_i)\Big]-a\delta_x^2 w_i^{\frac{1}{2}}+\Delta_x\delta_t v_i^{\frac{1}{2}}=g_1(x_i,t_0),\quad 1\le i\le m-1, \tag{3.4c}$$

$$\frac{2}{\tau}\Big[\delta_t v_i^{\frac{1}{2}}-\psi_2(x_i)\Big]-c\delta_x^2 v_i^{\frac{1}{2}}+b\Delta_x\delta_t w_i^{\frac{1}{2}}=g_2(x_i,t_0),\quad 1\le i\le m-1, \tag{3.4d}$$

$$w_i^0=\phi_1(x_i),\quad v_i^0=\psi_1(x_i),\quad 1\le i\le m-1, \tag{3.4e}$$

$$w_0^k=\alpha_1(t_k),\quad w_m^k=\alpha_2(t_k),\quad 0\le k\le n, \tag{3.4f}$$

$$v_0^k=\beta_1(t_k),\quad v_m^k=\beta_2(t_k),\quad 0\le k\le n. \tag{3.4g}$$

Remark 3.1. Since

$$\begin{aligned} u(x_i,t_k)&=u(x_i,t_0)+\int_{t_0}^{t_k} w(x_i,s)ds\\ &=u(x_i,t_0)+\tau\Big[\frac{1}{2}w(x_i,t_0)+\sum_{l=1}^{k-1}w(x_i,t_l)+\frac{1}{2}w(x_i,t_k)\Big]+\mathcal{O}(\tau^2), \end{aligned} \tag{3.5}$$

we let

$$u_i^k=u_i^0+\tau\Big(\frac{1}{2}w_i^0+\sum_{l=1}^{k-1}w_i^l+\frac{1}{2}w_i^k\Big),\quad 1\le i\le m-1,\quad 1\le k\le n, \tag{3.6}$$

when $\{w_i^k|0\le i\le m,\ 0\le k\le n\}$ has been determined.

Theorem 3.1. *The difference scheme* (3.4a)-(3.4g) *is uniquely solvable.*

Proof. The mathematical induction will be used to prove this theorem. The initial values $\{w_i^0,\ v_i^0|0\le i\le m\}$ are determined according to (3.4e)-(3.4g). Now, we consider the homogeneous system of (3.4c)-(3.4d) with (3.4f) and (3.4g)

$$\frac{2}{\tau^2}w_i^1-\frac{1}{2}a\delta_x^2 w_i^1+\frac{1}{\tau}\Delta_x v_i^1=0,\quad 1\le i\le m-1, \tag{3.7a}$$

$$\frac{2}{\tau^2}v_i^1-\frac{1}{2}c\delta_x^2 v_i^1+\frac{b}{\tau}\Delta_x w_i^1=0,\quad 1\le i\le m-1, \tag{3.7b}$$

$$w_0^1=0,\qquad w_m^1=0, \tag{3.7c}$$

$$v_0^1=0,\qquad v_m^1=0. \tag{3.7d}$$

Multiplying (3.7a) by hbw_i^1 and (3.7b) by hv_i^1, summing up for i from 1 to $m-1$, then adding the results, we obtain

$$\frac{2}{\tau^2}(b\|w^1\|^2+\|v^1\|^2)+\frac{1}{2}ab|w^1|_1^2+\frac{1}{2}c|v^1|_1^2=0,$$

which leads to

$$w_i^1=0,\quad v_i^1=0,\quad 0\le i\le m.$$

This implies that the difference scheme (3.4c)-(3.4d) with (3.4f)-(3.4g) uniquely determines $\{w_i^1,v_i^1|0\le i\le m\}$.

Then if $\{w_i^k,v_i^k|0\le i\le m\}$ and $\{w_i^{k-1},v_i^{k-1}|0\le i\le m\}$ have been obtained, the difference scheme (3.4a)-(3.4b) with (3.4f)-(3.4g) is a linear system of $\{w_i^{k+1},v_i^{k+1}|0\le i\le m\}$. Similarly, considering its homogeneous system, we have

$$\frac{1}{\tau^2}(b\|w^{k+1}\|+\|v^{k+1}\|^2)+\frac{1}{2}ab|w^{k+1}|_1^2+\frac{1}{2}c|v^{k+1}|_1^2=0.$$

Hence,

$$w_i^{k+1}=0,\quad v_i^{k+1}=0,\quad 0\le i\le m.$$

So the difference scheme (3.4a)-(3.4g) has a unique solution. This completes the proof. $\square$

4 The stability and convergence of difference scheme

In this section, we will prove the stability and convergence of the finite difference scheme (3.4a)-(3.4g). Firstly, as a counterpart of Theorem 2.1, we give a priori estimate of the difference scheme.

Lemma 4.1. *Assume that $\{w_i^k,v_i^k|0\le i\le m,\ 0\le k\le n\}$ is the solution of the following problem:*

$$\delta_t^2 w_i^k-\frac{a}{2}(\delta_x^2 w_i^{k+1}+\delta_x^2 w_i^{k-1})+\Delta_x\Delta_t v_i^k=(g_1)_i^k,\quad 1\le i\le m-1,\quad 1\le k\le n-1, \tag{4.1a}$$

$$\delta_t^2 v_i^k-\frac{c}{2}(\delta_x^2 v_i^{k+1}+\delta_x^2 v_i^{k-1})+b\Delta_x\Delta_t w_i^k=(g_2)_i^k,\quad 1\le i\le m-1,\quad 1\le k\le n-1, \tag{4.1b}$$

$$\frac{2}{\tau}\delta_t w_i^{\frac12}-a\delta_x^2 w_i^{\frac12}+\Delta_x\delta_t v_i^{\frac12}=(g_1)_i^0,\quad 1\le i\le m-1, \tag{4.1c}$$

$$\frac{2}{\tau}\delta_t v_i^{\frac12}-c\delta_x^2 v_i^{\frac12}+b\Delta_x\delta_t w_i^{\frac12}=(g_2)_i^0,\quad 1\le i\le m-1, \tag{4.1d}$$

$$w_i^0=\phi_1(x_i),\quad v_i^0=\psi_1(x_i),\quad 1\le i\le m-1, \tag{4.1e}$$

$$w_0^k=0,\quad w_m^k=0,\quad 0\le k\le n, \tag{4.1f}$$

$$v_0^k=0,\quad v_m^k=0,\quad 0\le k\le n, \tag{4.1g}$$

where $\phi_1(x_0)=\phi_1(x_m)=\psi_1(x_0)=\psi_1(x_m)=0$. Then we have

$$\begin{aligned}
&bh\sum_{i=1}^{m-1}(\delta_t w_i^{k+\frac{1}{2}})^2+h\sum_{i=1}^{m-1}(\delta_t v_i^{k+\frac{1}{2}})^2+\frac{ab}{2}(|w^{k+1}|_1^2+|w^k|_1^2)+\frac{c}{2}(|v^{k+1}|_1^2+|v^k|_1^2)\\
\le& e^{\frac{3}{2}k\tau}\Big[ab|\phi_1|_1^2+c|\psi_1|_1^2+\frac{b\tau^2}{4}\|(g_1)^0\|^2+\frac{\tau^2}{4}\|(g_2)^0\|^2\\
&+\frac{3}{2}\tau\sum_{l=1}^{k}\Big(b\|(g_1)^l\|^2+\|(g_2)^l\|^2\Big)\Big],\quad 0\le k\le n-1. \qquad (4.2)
\end{aligned}$$

If $(g_1)_i^k\equiv 0$, $(g_2)_i^k\equiv 0$, we have

$$\begin{aligned}
&bh\sum_{i=1}^{m-1}(\delta_t w_i^{k+\frac{1}{2}})^2+h\sum_{i=1}^{m-1}(\delta_t v_i^{k+\frac{1}{2}})^2+\frac{ab}{2}(|w^{k+1}|_1^2+|w^k|_1^2)+\frac{c}{2}(|v^{k+1}|_1^2+|v^k|_1^2)\\
=&bh\sum_{i=1}^{m-1}(\delta_t w_i^{\frac{1}{2}})^2+h\sum_{i=1}^{m-1}(\delta_t v_i^{\frac{1}{2}})^2+\frac{ab}{2}(|w^1|_1^2+|\phi_1|_1^2)+\frac{c}{2}(|v^1|_1^2+|\psi_1|_1^2),\quad 0\le k\le n-1. \quad (4.3)
\end{aligned}$$

Proof. Multiplying (4.1a) by $2bh\Delta_t w_i^k$ and summing up for i from 1 to $m-1$, it follows

$$\begin{aligned}
&2bh\sum_{i=1}^{m-1}(\delta_t^2 w_i^k)(\Delta_t w_i^k)-abh\sum_{i=1}^{m-1}(\delta_x^2 w_i^{k-1}+\delta_x^2 w_i^{k+1})\Delta_t w_i^k\\
&+2bh\sum_{i=1}^{m-1}(\Delta_x\Delta_t v_i^k)\Delta_t w_i^k=2bh\sum_{i=1}^{m-1}(g_1)_i^k\Delta_t w_i^k,\quad 1\le k\le n-1,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{b}{\tau}\Big[h\sum_{i=1}^{m-1}(\delta_t w_i^{k+\frac{1}{2}})^2-h\sum_{i=1}^{m-1}(\delta_t w_i^{k-\frac{1}{2}})^2\Big]+\frac{ab}{2\tau}(|w^{k+1}|_1^2-|w^{k-1}|_1^2)\\
&+2bh\sum_{i=1}^{m-1}(\Delta_x\Delta_t v_i^k)\Delta_t w_i^k=2bh\sum_{i=1}^{m-1}(g_1)_i^k\Delta_t w_i^k,\quad 1\le k\le n-1. \qquad (4.4)
\end{aligned}$$

Similarly, multiplying (4.1b) by $2h\Delta_t v_i^k$ and summing up for i from 1 to $m-1$, we can obtain

$$\begin{aligned}
&\frac{1}{\tau}\Big[h\sum_{i=1}^{m-1}(\delta_t v_i^{k+\frac{1}{2}})^2-h\sum_{i=1}^{m-1}(\delta_t v_i^{k-\frac{1}{2}})^2\Big]+\frac{c}{2\tau}(|v^{k+1}|_1^2-|v^{k-1}|_1^2)\\
&+2bh\sum_{i=1}^{m-1}(\Delta_x\Delta_t w_i^k)\Delta_t v_i^k=h\sum_{i=1}^{m-1}(g_2)_i^k\Delta_t v_i^k,\quad 1\le k\le n-1. \qquad (4.5)
\end{aligned}$$

Denote

$$E^k=bh\sum_{i=1}^{m-1}(\delta_t w_i^{k+\frac{1}{2}})^2+h\sum_{i=1}^{m-1}(\delta_t v_i^{k+\frac{1}{2}})^2+\frac{ab}{2}(|w^{k+1}|_1^2+|w^k|_1^2)+\frac{c}{2}(|v^{k+1}|_1^2+|v^k|_1^2).$$

Adding (4.4) and (4.5), we have

$$\begin{aligned}
&\frac{1}{\tau}(E^k-E^{k-1})+2bh\sum_{i=1}^{m-1}\Big[(\Delta_x\Delta_t v_i^k)\Delta_t w_i^k+(\Delta_x\Delta_t w_i^k)\Delta_t v_i^k\Big]\\
=&2bh\sum_{i=1}^{m-1}(g_1)_i^k\Delta_t w_i^k+h\sum_{i=1}^{m-1}(g_2)_i^k\Delta_t v_i^k,\quad 1\le k\le n-1.
\end{aligned}\tag{4.6}$$

For the second term on the left hand side of (4.6), using (4.1f) and (4.1g), we have

$$\begin{aligned}
&2bh\sum_{i=1}^{m-1}\Big[(\Delta_x\Delta_t v_i^k)\Delta_t w_i^k+(\Delta_x\Delta_t w_i^k)\Delta_t v_i^k\Big]\\
=&b\sum_{i=1}^{m-1}\Big[(\Delta_t v_{i+1}^k-\Delta_t v_{i-1}^k)\Delta_t w_i^k+(\Delta_t w_{i+1}^k-\Delta_t w_{i-1}^k)\Delta_t v_i^k\Big]\\
=&b\sum_{i=1}^{m-1}\Big[(\Delta_t v_{i+1}^k\Delta_t w_i^k+\Delta_t w_{i+1}^k\Delta_t v_i^k)-(\Delta_t v_i^k\Delta_t w_{i-1}^k+\Delta_t w_i^k\Delta_t v_{i-1}^k)\Big]\\
=&b\Big[(\Delta_t v_m^k\Delta_t w_{m-1}^k+\Delta_t w_m^k\Delta_t v_{m-1}^k)-(\Delta_t v_1^k\Delta_t w_0^k+\Delta_t w_1^k\Delta_t v_0^k)\Big]=0.
\end{aligned}$$

Consequently,

$$\frac{1}{\tau}(E^k-E^{k-1})=2bh\sum_{i=1}^{m-1}(g_1)_i^k\Delta_t w_i^k+h\sum_{i=1}^{m-1}(g_2)_i^k\Delta_t v_i^k,\quad 1\le k\le n-1.\tag{4.7}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\frac{1}{\tau}(E^k-E^{k-1})\le&\frac{b}{2}h\sum_{i=1}^{m-1}\Big[(\delta_t w_i^{k+\frac12})^2+(\delta_t w_i^{k-\frac12})^2\Big]+\frac12 h\sum_{i=1}^{m-1}\Big[(\delta_t v_i^{k+\frac12})^2\\
&+(\delta_t v_i^{k-\frac12})^2\Big]+b\|(g_1)^k\|^2+\|(g_2)^k\|^2\\
\le&\frac12\big(E^k+E^{k-1}\big)+b\|(g_1)^k\|^2+\|(g_2)^k\|^2,\quad 1\le k\le n-1,
\end{aligned}$$

or,

$$\Big(1-\frac{\tau}{2}\Big)E^k\le\Big(1+\frac{\tau}{2}\Big)E^{k-1}+\tau\Big(b\|(g_1)^k\|^2+\|(g_2)^k\|^2\Big),\quad 1\le k\le n-1.$$

When $\tau\le 2/3$, it follows

$$E^k\le\Big(1+\frac{3\tau}{2}\Big)E^{k-1}+\frac32\tau\Big(b\|(g_1)^k\|^2+\|(g_2)^k\|^2\Big),\quad 1\le k\le n-1.$$

Using the Gronwall inequality yields

$$E^k\le e^{\frac32 k\tau}\Big[E^0+\frac32\tau\sum_{l=1}^{k}\Big(b\|(g_1)^l\|^2+\|(g_2)^l\|^2\Big)\Big],\quad 1\le k\le n-1.\tag{4.8}$$

Multiplying (4.1c) by $bh\delta_t w_i^{1/2}$ and (4.1d) by $h\delta_t v_i^{1/2}$, respectively, then adding the results and summing up for i from 1 to $m-1$, we obtain

$$
\begin{aligned}
&b\|\delta_t w^{\frac{1}{2}}\|^2+\|\delta_t v^{\frac{1}{2}}\|^2+\frac{ab}{2}(|w^1|_1^2+|w^0|_1^2)+\frac{c}{2}(|v^1|_1^2+|v^0|_1^2)\\
\le& ab|w^0|_1^2+c|v^0|_1^2+\frac{b\tau^2}{4}\|(g_1)^0\|^2+\frac{\tau^2}{4}\|(g_2)^0\|^2.
\end{aligned}
$$

Noticing the definition of E^k and inserting the inequality above into (4.8), we can obtain (4.2). When $(g_1)_i^k\equiv 0$, $(g_2)_i^k\equiv 0$, (4.7) implies (4.3). This completes the proof. □

Using Lemma 4.1, we can obtain the following result on the stability.

Theorem 4.1. *The difference scheme* (3.4a)-(3.4g) *is stable with respect to the initial values and right-hand side functions.*

Now we present the convergence result.

Theorem 4.2. *Assume that the solution of* (1.4a)-(1.4f) *is sufficiently smooth. The difference scheme* (3.4a)-(3.4g) *is convergent with the convergence order of two both in time and space in the maximum norm. More precisely, denote*

$$e_i^k=W_i^k-w_i^k,\quad f_i^k=V_i^k-v_i^k,\quad 0\le i\le m,\quad 0\le k\le n.$$

Then the following estimates hold

$$\|e^k\|_\infty\le\frac{c_0}{2}e^{\frac{3}{4}T}\sqrt{\frac{(1+6T)(b+1)}{2ab}}(\tau^2+h^2),\qquad 1\le k\le n, \tag{4.9a}$$

$$\|f^k\|_\infty\le\frac{c_0}{2}e^{\frac{3}{4}T}\sqrt{\frac{(1+6T)(b+1)}{2c}}(\tau^2+h^2),\qquad 1\le k\le n. \tag{4.9b}$$

Proof. Subtracting (3.4a)-(3.4g) from (3.1a)-(3.1d) and (3.3a)-(3.3c), we have the error system:

$$\delta_t^2 e_i^k-\frac{a}{2}(\delta_x^2 e_i^{k+1}+\delta_x^2 e_i^{k-1})+\Delta_x\Delta_t f_i^k=(R_1)_i^k,\qquad 1\le i\le m-1,\quad 1\le k\le n-1, \tag{4.10a}$$

$$\delta_t^2 f_i^k-\frac{c}{2}(\delta_x^2 f_i^{k+1}+\delta_x^2 f_i^{k-1})+b\Delta_x\Delta_t e_i^k=(R_2)_i^k,\qquad 1\le i\le m-1,\quad 1\le k\le n-1, \tag{4.10b}$$

$$\frac{2}{\tau}\delta_t e_i^{\frac{1}{2}}-a\delta_x^2 e_i^{\frac{1}{2}}+\Delta_x\delta_t f_i^{\frac{1}{2}}=(R_1)_i^0,\quad 1\le i\le m-1, \tag{4.10c}$$

$$\frac{2}{\tau}\delta_t f_i^{\frac{1}{2}}-c\delta_x^2 f_i^{\frac{1}{2}}+b\Delta_x\delta_t w_i^{\frac{1}{2}}=(R_2)_i^0,\quad 1\le i\le m-1, \tag{4.10d}$$

$$e_i^0=0,\quad f_i^0=0,\quad 1\le i\le m-1, \tag{4.10e}$$

$$e_0^k=0,\quad e_m^k=0,\quad 0\le k\le n, \tag{4.10f}$$

$$f_0^k=0,\quad f_m^k=0,\quad 0\le k\le n. \tag{4.10g}$$

Utilizing Lemma 4.1, we have

$$\begin{aligned}
&bh\sum_{i=1}^{m-1}(\delta_t e_i^{k+\frac{1}{2}})^2+h\sum_{i=1}^{m-1}(\delta_t f_i^{k+\frac{1}{2}})^2+\frac{ab}{2}(|e^{k+1}|_1^2+|e^k|_1^2)+\frac{c}{2}(|f^{k+1}|_1^2+|f^k|_1^2)\\
\le& e^{\frac{3}{2}k\tau}\Big[ab|e^0|_1^2+c|f^0|_1^2+\frac{b\tau^2}{4}\|(R_1)^0\|^2+\frac{\tau^2}{4}\|(R_2)^0\|^2\\
&+\frac{3}{2}\tau\sum_{l=1}^{k}(b\|(R_1)^l\|^2+\|(R_2)^l\|^2)\Big],\quad 0\le k\le n-1.
\end{aligned}$$

Substituting (3.2a)-(3.2b) into the above inequality, it yields

$$\begin{aligned}
&bh\sum_{i=1}^{m-1}(\delta_t e_i^{k+\frac{1}{2}})^2+h\sum_{i=1}^{m-1}(\delta_t f_i^{k+\frac{1}{2}})^2+\frac{ab}{2}\Big(|e^{k+1}|_1^2+|e^k|_1^2\Big)+\frac{c}{2}(|f^{k+1}|_1^2+|f^k|_1^2)\\
\le& e^{\frac{3}{2}T}\Big[\Big(\frac{1}{4}+\frac{3}{2}T\Big)(1+b)\Big]c_0^2(\tau^2+h^2)^2,\quad 0\le k\le n-1. \qquad (4.11)
\end{aligned}$$

It completes the proof by applying the discrete Sobolev inequalities $\|e^k\|_\infty\le|e^k|_1/2$ and $\|f^k\|_\infty\le|f^k|_1/2$ to (4.11). $\square$

Remark 4.1. It follows from (3.5) and (3.6) that

$$\begin{aligned}
u(x_i,t_k)-u_i^k=&\tau\Big[\frac{1}{2}\Big(w(x_i,t_0)-w_i^0\Big)+\sum_{l=1}^{k-1}\Big(w(x_i,t_l)-w_i^l\Big)\\
&+\frac{1}{2}\Big(w(x_i,t_k)-w_i^k\Big)\Big]+\mathcal{O}(\tau^2),\quad 1\le i\le m-1,\quad 1\le k\le n.
\end{aligned}$$

Using Theorem 4.2, we obtain

$$\max_{\substack{1\le i\le m-1\\ 1\le k\le n}}|u(x_i,t_k)-u_i^k|=\mathcal{O}(\tau^2+h^2).$$

5 The computation of the difference scheme

Denote $W^k=(w_1^k,w_2^k,\cdots,w_{m-1}^k)^T$ and $V^k=(v_1^k,v_2^k,\cdots,v_{m-1}^k)^T$. $\{W^0,V^0\}$ are given by (3.4f). The difference scheme (3.4c)-(3.4d) with (3.4f)-(3.4g) is a linear system of algebraic equations about unknowns $\{W^1,V^1\}$. Suppose $\{W^{k-1},V^{k-1}\}$ and $\{W^k,V^k\}$ have been determined. The difference scheme (3.4a)-(3.4b) with (3.4f)-(3.4g) is a linear system of algebraic equations about unknowns $\{W^{k+1},V^{k+1}\}$.

We write (3.4a)-(3.4b) in matrix form as follows.

Denote $s=\tau/h$, and let $S_i^k=\begin{pmatrix} w_i^k\\ v_i^k\end{pmatrix}$. We can write (3.4a)-(3.4b) in the form

$$AS_{i-1}^{k+1}+BS_i^{k+1}+CS_{i+1}^{k+1}=J_i^k,\quad 1\le i\le m-1, \qquad (5.1)$$

where

$$A=\begin{bmatrix} -\frac{a}{2}s^2 & -\frac{1}{4}s \\ -\frac{b}{4}s & -\frac{c}{2}s^2 \end{bmatrix},\quad B=\begin{bmatrix} 1+as^2 & 0 \\ 0 & 1+cs^2 \end{bmatrix},\quad C=\begin{bmatrix} -\frac{a}{2}s^2 & \frac{1}{4}s \\ \frac{b}{4}s & -\frac{c}{2}s^2 \end{bmatrix},$$

and $J_i^k \in R^2$ is known. Noticing (3.4f)-(3.4g), (5.1) may be rewritten as the following matrix form:

$$\begin{bmatrix} B & C & 0 & \cdots & 0 & 0 \\ A & B & C & \cdots & C & 0 \\ 0 & A & B & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B & C \\ 0 & 0 & 0 & \cdots & A & B \end{bmatrix}_{(2m-2)\times(2m-2)} \begin{bmatrix} S_1^{k+1} \\ S_2^{k+1} \\ \vdots \\ S_{m-2}^{k+1} \\ S_{m-1}^{k+1} \end{bmatrix}_{(2m-2)\times 1}$$

$$=\begin{bmatrix} J_1^k - AS_0^{k+1} \\ J_2^k \\ \vdots \\ J_{m-2}^k \\ J_{m-1}^k - CS_m^{k+1} \end{bmatrix}_{(2m-2)\times 1}.$$

This is a tri-diagonal block system of linear algebraic equations, which can be solved by double-sweep method.

Moreover, let

$$A_1=\text{tridiag}\Big(-\frac{a}{2}s^2, 1+as^2, -\frac{a}{2}s^2\Big)_{m-1}, \qquad B_1=\text{tridiag}\Big(-\frac{1}{4}s, 0, \frac{1}{4}s\Big)_{m-1},$$

$$A_2=\text{tridiag}\Big(-\frac{c}{2}s^2, 1+cs^2, -\frac{c}{2}s^2\Big)_{m-1}, \qquad B_2=\text{tridiag}\Big(-\frac{b}{4}s, 0, \frac{b}{4}s\Big)_{m-1},$$

and noticing (3.4f)-(3.4g), the difference scheme (3.4a)-(3.4b) can be written in the following form:

$$\begin{bmatrix} A_1 & B_1 \\ B_2 & A_2 \end{bmatrix}\begin{bmatrix} W^{k+1} \\ V^{k+1} \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{5.2}$$

where $f_1 \in R^{m-1}$, $f_2 \in R^{m-1}$, which depend on $\{W^{k-1}, V^{k-1}\}$ and $\{W^k, V^k\}$. Thus at the $(k+1)$-th time level, the system of linear algebraic equations about the wanted solution $\{W^{k+1}, V^{k+1}\}$ can be obtained by the iterative method.

Similarly, we can also write (3.4c)-(3.4d) in the matrix form.

6 Numerical experiment

In this section, we give the following example to verify the theoretical results obtained in the previous section.

Consider the following problem:

$$
\begin{aligned}
& w_{tt}-w_{xx}+v_{xt}=g_1(x,t), && 0<x<1,\quad 0<t\le 1, && \text{(6.1a)}\\
& v_{tt}-4v_{xx}+w_{xt}=g_2(x,t), && 0<x<1,\quad 0<t\le 1, && \text{(6.1b)}\\
& w(x,0)=1,\quad w_t(x,0)=x, && 0\le x\le 1, && \text{(6.1c)}\\
& v(x,0)=e,\quad v_t(x,0)=0, && 0\le x\le 1, && \text{(6.1d)}\\
& w(0,t)=1,\quad w(1,t)=e^{\sin t}, && 0<t\le 1, && \text{(6.1e)}\\
& v(0,t)=e,\quad v(1,t)=e^{\cos t}, && 0<t\le 1, && \text{(6.1f)}
\end{aligned}
$$

where

$$
\begin{aligned}
g_1(x,t)=&e^{\sin(xt)}\left[\sin(xt)-\cos^2(xt)\right](t^2-x^2)\\
&+e^{\cos(xt)}\left[xt\sin^2(xt)-\sin(xt)-xt\cos(xt)\right],\\
g_2(x,t)=&e^{\cos(xt)}\left[\cos(xt)-\sin^2(xt)\right](4t^2-x^2)\\
&+e^{\sin(xt)}\left[\cos(xt)-xt\sin(xt)+xt\cos^2(xt)\right].
\end{aligned}
$$

Its exact solutions are $w(x,t)=e^{\sin(xt)}$ and $v(x,t)=e^{\cos(xt)}$. Let $h=1/m$ and $\tau=h$. We utilize the difference scheme (3.4a)-(3.4g) to compute the numerical solution of the problem (6.1a)-(6.1f) with the help of the Gauss-Seidel iterative method.

Some numerical values at the selected points computed with different step sizes are listed in Table 1 and Table 2. Correspondingly, the absolute errors at these points are shown in Table 3 and Table 4. In addition, the error curves are shown in Fig. 1 and Fig. 2.

Table 1: Some numerical results of w at $t=1$.

$M\backslash x$	0.125	0.375	0.625	0.875
8	1.1274383	1.4360708	1.7923393	2.1540602
16	1.1313092	1.4401695	1.7949539	2.1545294
32	1.1324077	1.4417774	1.7951267	2.1544920
64	1.1326869	1.4422037	1.7951571	2.1544737
128	1.1327567	1.4423123	1.7951633	2.1544690
256	1.1327741	1.4423393	1.7951650	2.1544677
$w(x,1)$	1.1327799	1.4423483	1.7951656	2.1544673

Table 2: Some numerical results of v at $t=1$.

$M\backslash x$	0.125	0.375	0.625	0.875
8	2.7005306	2.5439329	2.2594409	1.9031193
16	2.6979712	2.5379248	2.2524871	1.8995434
32	2.6973573	2.5363178	2.2506602	1.8986576
64	2.6972030	2.5359267	2.2502203	1.8984399
128	2.6971671	2.5358286	2.2501104	1.8983890
256	2.6971583	2.5358042	2.2500831	1.8983765
$v(x,1)$	2.6971554	2.5357961	2.2500740	1.8983723

Table 3: The absolute errors of numerical solutions of w at $t=1$.

$M\backslash x$	0.125	0.375	0.625	0.875
8	5.342e-03	6.277e-03	2.826e-03	4.071e-04
16	1.471e-03	2.179e-03	2.117e-04	6.212e-05
32	3.722e-04	5.709e-04	3.890e-05	2.471e-05
64	9.300e-05	1.445e-04	8.481e-06	6.394e-06
128	2.329e-05	3.598e-05	2.264e-06	1.675e-06
256	5.817e-06	9.004e-06	5.583e-07	4.182e-07

Table 4: The absolute errors of numerical solutions of v at $t=1$.

$M\backslash x$	0.125	0.375	0.625	0.875
8	3.375e-03	8.137e-03	9.367e-03	4.747e-03
16	8.158e-04	2.129e-03	2.413e-03	1.171e-03
32	2.019e-04	5.217e-04	5.861e-04	2.852e-04
64	4.762e-05	1.306e-04	1.463e-04	6.760e-05
128	1.168e-05	3.251e-05	3.641e-05	1.661e-05
256	2.905e-06	8.128e-06	9.100e-06	4.133e-06

Table 5: The max errors of the numerical solutions of w and v.

M	$E_\infty(h)$	$E_\infty(2h)/E_\infty(h)$	$F_\infty(h)$	$F_\infty(2h)/F_\infty(h)$
8	7.184e-03	*	9.367e-03	*
16	2.378e-03	3.0214	2.470e-03	3.7925
32	6.252e-04	3.8032	5.964e-04	4.1415
64	1.546e-04	4.0436	1.501e-04	3.9740
128	3.875e-05	3.9903	3.760e-05	3.9907
256	9.693e-06	3.9977	9.409e-06	3.9967

We define the maximum errors as follows:

$$E_\infty(h)=\max_{1\le i\le m-1,1\le k\le n}|w(x_i,t_k)-w_i^k|,\quad F_\infty(h)=\max_{1\le i\le m-1,1\le k\le n}|v(x_i,t_k)-v_i^k|.$$

The maximum errors and the convergence orders are listed in Table 5.

Suppose

$$E_\infty(h)\approx c_1h^{r_1},\quad F_\infty(h)\approx c_2h^{r_2},$$

then

$$\log E_\infty(h)\approx\log c_1+r_1(\log h),\quad \log F_\infty(h)\approx\log c_2+r_2(\log h).$$

Using the data in Table 5, we obtain

$$\log E_\infty(h)\approx -0.3414+1.9286(\log h),\quad \log F_\infty(h)\approx -0.2160+1.9971(\log h).$$

From these tables and figures, we conclude that the errors of the difference solution in maximal norm decreases by a factor of 4 as the mesh sizes are reduced by a factor of 2, which are in great accordance with our theoretical results.

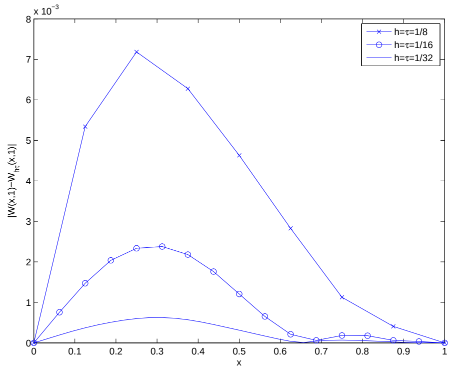

Figure 1: Error curves of numerical solutions of w at $t=1$.

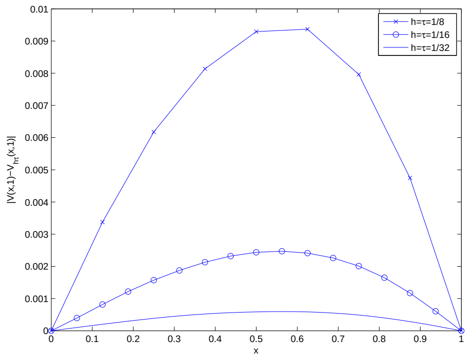

Figure 2: Error curves of numerical solutions of v at $t=1$.

7 Conclusions

In this article, the numerical solution to a magneto-thermo-elasticity model was considered. At first, the model was reformulated into an alternate form by introducing a new function. Then the priori estimate of the solution was presented. A difference scheme was established and the unique solvability, unconditional stability and convergence were analyzed. The convergence order in maximum norm are two in both time and in space. Next, two matrix forms of the difference scheme were given. The double sweep method and the iterative method were used to solve the difference scheme. Finally, one numerical example was presented to support the theoretical results.

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