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An Inverse Problem of Determining Coefficients in a One-Dimensional Radiative Transport Equation

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Abstract. We consider an inverse problem of determining unknown coefficients for a one-dimensional analogue of radiative transport equation. We show that some combination of the unknown coefficients can be uniquely determined by giving pulse-like inputs at the boundary and observing the corresponding outputs. Our result can be applied for determination of absorption and scattering properties of an optically turbid medium if the radiative transport equation is appropriate for describing the propagation of light in the medium.

AMS subject classifications: 35R30, 35L50, 35Q99, 62P10

Key words: Inverse problem, radiative transport equation, first-order hyperbolic system, optical tomography.

1 Introduction

We consider a one-dimensional version of the inverse problem of identifying unknown coefficients of the one-speed, time-dependent radiative transport equation. This inverse problem is related with the study of optical tomography (see, e.g., [1, 2] and references therein). Optical tomography has been studied for several decades as a new modality of medical imaging technique using low-energy light in the near-infrared region. Compared with other tomographic techniques using high-energy radiation (e.g., X-ray CT), optical tomography is considered to be less harmful to human body. In most researches on optical tomography the propagation of near-infrared light in biological tissues is modeled by the radiative transport equation, and the process of imaging is formulated as an inverse problem of determining unknown coefficients of the equation. Although the original problem is three-dimensional in space variables, our discussion here is limited to the

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one-dimensional case partly because the three-dimensional problem is quite difficult and mainly because we can obtain a reconstruction formula for the unknown coefficients.

Let $(0,H)=\{x\in\mathbb{R};\, 0<x<H\}$ be a finite open interval in $\mathbb{R}$. Let $\mu_{\rm a}(x)$, $\mu_{\rm s}(x)$, and $q(x)$ be continuous functions on the closed interval $[0,H]$ and assume that $\mu_{\rm a}(x)\ge 0$, $\mu_{\rm s}(x)\ge 0$, and $0\le q(x)\le 1$ there. We consider the situation where the interval $[0,H]$ is occupied by a medium that absorbs and scatters photons, with $\mu_{\rm a}$, $\mu_{\rm s}$, and q being the distribution of the absorption coefficient, the scattering coefficient, and the probability of backward scattering, respectively. Let $I_1(x,t)$ be the density of photons moving through the medium with speed c in the positive x direction, and $I_2(x,t)$ in the negative x direction. The time evolution of I_1 and I_2 are described by the system of differential equations

$$\frac{1}{c}\frac{\partial I_1}{\partial t}+\frac{\partial I_1}{\partial x}=-(\mu_{\rm a}+q\mu_{\rm s})I_1+q\mu_{\rm s}I_2,\quad 0<x<H,\quad 0<t<T, \tag{1.1a}$$

$$\frac{1}{c}\frac{\partial I_2}{\partial t}-\frac{\partial I_2}{\partial x}=-(\mu_{\rm a}+q\mu_{\rm s})I_2+q\mu_{\rm s}I_1,\quad 0<x<H,\quad 0<t<T, \tag{1.1b}$$

where c and T are positive numbers. We always assign the initial condition

$$I_1(x,0)=I_2(x,0)=0,\quad 0\le x\le H. \tag{1.2}$$

We assume that the speed c is a known constant, while the coefficients $\mu_{\rm a}(x)$, $\mu_{\rm s}(x)$, and $q(x)$ are unknown functions. In order to determine those unknowns, we consider an experiment as follows. We give a pulse-like input at one end of the interval $[0,H]$ and observe the boundary values of the outward flow at both ends, i.e., $I_1(H,t)$ and $I_2(0,t)$. We again follow the same process by giving the input at the other end. To be precise, we solve (1.1) and (1.2) with the boundary condition

$$I_1(0,t)=\delta(t),\quad I_2(H,t)=0. \tag{1.3}$$

Writing the solution to (1.1), (1.2), and (1.3) as $I^1=(I^1_1,I^1_2)$, we observe

$$I^1_1(H,t),\quad I^1_2(0,t),\quad 0\le t\le T. \tag{1.4}$$

Next we solve (1.1) and (1.2) with the boundary condition

$$I_1(0,t)=0,\quad I_2(H,t)=\delta(t), \tag{1.5}$$

write the solution to (1.1), (1.2), and (1.5) as $I^2=(I^2_1,I^2_2)$, and then observe

$$I^2_1(H,t),\quad I^2_2(0,t),\quad 0\le t\le T. \tag{1.6}$$

Our main result is as follows.

Theorem 1.1. *Let m and M be positive numbers with $m<M$. In the setting above, we consider the admissible set $A(m,M)$ of the unknown coefficients satisfying*

$$\mu_a,\ \mu_s,\ q\in C^0[0,H]\quad \text{and}\quad \mu_a,\mu_s\ge 0,\quad 0\le q\le 1,\quad m\le \mu_a+q\mu_s\le M\ \text{ on }\ [0,H].$$

Then there exists a number $H^*>0$, such that if $0<H<H^*$, $T\ge 2H/c$, and $(\mu_a,\mu_s,q)\in A(m,M)$, the data (1.4) *and* (1.6) *uniquely identify μ_a and $q\mu_s$.*