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Explicit Multi-Symplectic Splitting Methods for the Nonlinear Dirac Equation

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Abstract. In this paper, we propose two new explicit multi-symplectic splitting methods for the nonlinear Dirac (NLD) equation. Based on its multi-symplectic formulation, the NLD equation is split into one linear multi-symplectic system and one nonlinear infinite Hamiltonian system. Then multi-symplectic Fourier pseudospectral method and multi-symplectic Preissmann scheme are employed to discretize the linear subproblem, respectively. And the nonlinear subsystem is solved by a symplectic scheme. Finally, a composition method is applied to obtain the final schemes for the NLD equation. We find that the two proposed schemes preserve the total symplecticity and can be solved explicitly. Numerical experiments are presented to show the effectiveness of the proposed methods.

AMS subject classifications: 65M06, 65M70

Key words: Nonlinear Dirac equation, multi-symplectic method, splitting method, explicit method.

1 Introduction

In this paper, we consider the $(1+1)$-dimensional nonlinear Dirac (NLD) equation [1]

$$\begin{cases} \Psi_t = A\Psi_x + \mathrm{i}f(|\Psi_1|^2 - |\Psi_2|^2)B\Psi, \\ \Psi_1(x,0) = \phi_1(x), \quad \Psi_2(x,0) = \phi_2(x), \end{cases} \tag{1.1}$$

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where $\Psi=[\Psi_1,\Psi_2]^T$ is a spinorial wave function, which describes a particle with the spin $-1/2$. Here, Ψ_1 and Ψ_2 are complex functions, $\mathrm{i}=\sqrt{-1}$ is the imaginary unit, $f(s)$ is a real function of a real variable s, A and B are matrices

$$A=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Some numerical methods have been developed to solve the NLD equation (1.1), such as spectral methods [2] and finite difference methods [3–5]. In [6], finite volume methods with fine meshes are proposed to study the interaction dynamics of the Dirac solitary waves. In [7–9], high-order accurate Runge-Kutta discontinuous Galerkin method is also developed to simulate the solitary wave interaction of the NLD equation. More recently, an integrating-factor method for the NLD equation is proposed in [10]. In this paper, we aim to study efficient multi-symplectic methods for the NLD equation. Multi-symplectic methods are a kind of methods which can preserve the multi-symplectic conservation law of Hamiltonian partial differential equations (PDEs) under appropriate discretizations and perform better than tranditional methods in long time simulation [11], like the well known symplectic methods (see for instance [12–15]). Recently, such kind of methods have been paid a lot of attentions to [16]. Some multi-symplectic methods have been developed for the Hamiltonian PDEs, such as multi-symplectic Preissmann scheme [11, 17], multi-symplectic Runge-Kutta methods [18], multi-symplectic spectral discretizations [19], multi-symplectic Fourier pseudospectral method [20, 21], multi-symplectic wavelet collocation method [22–25], and so on. However, most of the multi-symplectic methods are implicit and not efficient enough in computation. In order to solve these problems, some efforts have also been made. In [26], splitting method is firstly introduced to reduce the difficulty of solving multi-symplectic methods. The effectiveness of multi-symplectic splitting methods is shown numerically in [27–29]. Using symplectic Runge-Kutta-Nyström methods and symplectic Runge-Kutta-type methods, Hong et al. developed explicit multi-symplectic methods for the wave equation [31] and the Klein-Gordon-Schrödinger equation [32], respectively.

In [33], it is shown that the NLD equation can be written into a multi-symplectic form. And based on such a formulation, multi-symplectic Runge-Kutta (MSRK) methods for the NLD equation are theoretically investigated. Furthermore, numerical experiments are presented to show the effectiveness of the MSRK methods for the NLD equation in [1]. However, the MSRK methods for the NLD equation are implicit. It is required to use a fixed-point iteration method to solve nonlinear equations which will cost a lot of efforts. In this paper, we develop two explicit multi-symplectic splitting methods for the NLD equation. Firstly, the NLD equation is split into one linear subproblem and one nonlinear subproblem. And then, the two subproblems are integrated separately. On the one hand, the linear subproblem is written as a multi-symplectic form. Then, multi-symplectic Fourier pseudospectral method and multi-symplectic Preissmann method are used to discretize this linear subproblem. Moreover, it is shown that the two proposed methods for the linear subproblem can be solved explicitly. On the other hand, the nonlinear subsystem for the NLD equation can be written as an infinite Hamiltonian system and solved explicitly by a symplectic scheme. Finally, the Strang splitting method is introduced to obtain the final explicit schemes, which preserve the total symplecticity of the NLD equation. In order to show the effectiveness of the proposed methods, numerical experiments of propagation and interaction of solitary wave solutions are presented. In addition, the preserving properties of conservation laws of charge, energy and momentum are also investigated.

The rest of this paper is arranged as follows. In the rest of this section, we give the definition of charge, energy and momentum of the NLD equation, and then consider an important case of (1.1). The conservation laws of charge, energy and momentum are also restated. In Section 2, multi-symplectic formulation of the NLD equation and its corresponding conservation laws are introduced. In Section 3, two kinds of multi-symplectic splitting methods are constructed for the NLD equation. Furthermore, it is shown that such methods can be solved explicitly. Numerical experiments are presented to show the effectiveness of the proposed methods in Section 4. Finally, conclusions are made in Section 5.

In this context, the charge $\mathcal{Q}$, the momentum $\mathcal{P}$ and the energy $\mathcal{E}$ of the NLD equation (1.1) are given by [1]

$$\begin{cases} \mathcal{Q}(\Psi)(t)=\int_{\mathbb{R}}\left(|\Psi_1(x,t)|^2+|\Psi_2(x,t)|^2\right)dx,\\ \mathcal{P}(\Psi)(t)=\int_{\mathbb{R}}\mathrm{Im}\Big(\overline{\Psi}_1\frac{\partial}{\partial x}\Psi_1+\overline{\Psi}_2\frac{\partial}{\partial x}\Psi_2\Big)dx,\\ \mathcal{E}(\Psi)(t)=\int_{\mathbb{R}}\Big(\mathrm{Im}\Big(\overline{\Psi}_1\frac{\partial}{\partial x}\Psi_2+\overline{\Psi}_2\frac{\partial}{\partial x}\Psi_1\Big)+\widetilde{f}(|\Psi_1|^2-|\Psi_2|^2)\Big)dx, \end{cases} \tag{1.2}$$

where $\mathrm{Im}(\Psi)$ and $\overline{\Psi}$ denote respectively the imaginary part and the conjugate of the complex Ψ, $\widetilde{f}$ is defined by $\widetilde{f}(s)=\int_0^s f(\tau)d\tau$.

In this paper, we consider an important case of the NLD equation (1.1) as that in [1]

$$\begin{cases} \dfrac{\partial\Psi_1}{\partial t}+\dfrac{\partial\Psi_2}{\partial x}+\mathrm{i}m\Psi_1+2\mathrm{i}\lambda(|\Psi_2|^2-|\Psi_1|^2)\Psi_1=0,\\ \dfrac{\partial\Psi_2}{\partial t}+\dfrac{\partial\Psi_1}{\partial x}-\mathrm{i}m\Psi_2+2\mathrm{i}\lambda(|\Psi_1|^2-|\Psi_2|^2)\Psi_2=0, \end{cases} \tag{1.3}$$

namely, $f(s)=m-2\lambda s$ in (1.1), where m and λ are real constants.

From [1], we know that the NLD equation (1.3) has the following conservation laws.

Proposition 1.1. If the solution Ψ of the NLD equation (1.3) satisfies

$$\lim_{|x|\to+\infty}|\Psi(x,t)|=0 \quad\text{and}\quad \lim_{|x|\to+\infty}|\partial_x\Psi(x,t)|=0 \ \text{ uniformly for } \ t\in\mathbb{R}, \tag{1.4}$$

then

$$\frac{d}{dt}\mathcal{Q}(\Psi)(t)=0,\quad \frac{d}{dt}\mathcal{P}(\Psi)(t)=0 \quad\text{and}\quad \frac{d}{dt}\mathcal{E}(\Psi)(t)=0.$$

2 Multi-symplectic formulation of the nonlinear Dirac equation

In this section, we restate some results from [1]. By letting $\Psi_1=p_1+\mathrm{i}q_1$, $\Psi_2=p_2+\mathrm{i}q_2$, the NLD equation (1.3) can be written as a system of real-value equations

$$\begin{cases} \dfrac{\partial p_1}{\partial t}+\dfrac{\partial p_2}{\partial x}-mq_1-2\lambda(p_2^2+q_2^2-p_1^2-q_1^2)q_1=0,\\ \dfrac{\partial q_1}{\partial t}+\dfrac{\partial q_2}{\partial x}+mp_1+2\lambda(p_2^2+q_2^2-p_1^2-q_1^2)p_1=0,\\ \dfrac{\partial p_2}{\partial t}+\dfrac{\partial p_1}{\partial x}+mq_2+2\lambda(p_2^2+q_2^2-p_1^2-q_1^2)q_2=0,\\ \dfrac{\partial q_2}{\partial t}+\dfrac{\partial q_1}{\partial x}-mp_2-2\lambda(p_2^2+q_2^2-p_1^2-q_1^2)p_2=0. \end{cases} \tag{2.1}$$

Moreover, the system (2.1) can be written as a multi-symplectic Hamiltonian PDE [11]

$$Mz_t+Kz_x=\nabla_z S(z) \tag{2.2}$$

with

$$z=[p_1,q_1,p_2,q_2]^{\mathrm{T}}, \qquad S(z)=\frac{1}{2}(\lambda(p_1^2+q_1^2-p_2^2-q_2^2)-m)(p_1^2+q_1^2-p_2^2-q_2^2),$$
$$M=\begin{bmatrix} J & 0\\ 0 & J\end{bmatrix}, \qquad K=\begin{bmatrix} 0 & J\\ J & 0\end{bmatrix}.$$

Here

$$J=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}.$$

From the multi-symplectic theories [11, 18], we know that the system (2.2) satisfies a multi-symplectic conservation law

$$\omega_t+\kappa_x=0, \tag{2.3}$$

where ω and κ are pre-symplectic forms with

$$\omega=\frac{1}{2}dz\wedge Mdz, \quad \kappa=\frac{1}{2}dz\wedge Kdz.$$

The system (2.2) also has a local energy conservation law

$$E_t+F_x=0 \;\text{ with }\; E(z)=S(z)-\frac{1}{2}z^{\mathrm{T}}Kz_x, \quad F(z)=\frac{1}{2}z^{\mathrm{T}}Kz_t, \tag{2.4}$$

and a momentum conservation law

$$I_t+G_x=0 \;\text{ with }\; G(z)=S(z)-\frac{1}{2}z^{\mathrm{T}}Mz_t, \quad I(z)=\frac{1}{2}z^{\mathrm{T}}Mz_x. \tag{2.5}$$

For the NLD equation (1.3), the above conservation laws are given explicitly by

$$\begin{aligned}
&\omega = dp_1\wedge dq_1 + dp_2\wedge dq_2, \quad \kappa = dp_1\wedge dq_2 + dp_2\wedge dq_1,\\
&E(z) = S(z) - \frac{1}{2}\Big(p_2\frac{\partial}{\partial x}q_1 - q_2\frac{\partial}{\partial x}p_1 + p_1\frac{\partial}{\partial x}q_2 - q_1\frac{\partial}{\partial x}p_2\Big),\\
&F(z) = \frac{1}{2}\Big(p_2\frac{\partial}{\partial x}q_1 - q_2\frac{\partial}{\partial x}p_1 + p_1\frac{\partial}{\partial x}q_2 - q_1\frac{\partial}{\partial x}p_2\Big),\\
&G(z) = S(z) - \frac{1}{2}\sum_{i=1}^{2}\Big(p_i\frac{\partial}{\partial t}q_i - q_i\frac{\partial}{\partial t}p_i\Big), \quad I(z) = \frac{1}{2}\sum_{i=1}^{2}\Big(p_i\frac{\partial}{\partial x}q_i - q_i\frac{\partial}{\partial x}p_i\Big).
\end{aligned}$$

In addition, for the NLD equation (1.3), it follows that

$$\mathrm{Im}\Big(\overline{\Psi}_1\frac{\partial}{\partial x}\Psi_2 + \overline{\Psi}_2\frac{\partial}{\partial x}\Psi_1\Big) + \widetilde{f}(|\Psi_1|^2 - |\Psi_2|^2) = -2E.$$

Hence,

$$\mathcal{E}(\Psi)(t) = -2\int_{\mathbb{R}} E(z(x,t))dx. \tag{2.6}$$

Similarly, we can obtain

$$\mathcal{P}(\Psi)(t) = \int_{\mathbb{R}} \mathrm{Im}\Big(\overline{\Psi}_1\frac{\partial}{\partial x}\Psi_1 + \overline{\Psi}_2\frac{\partial}{\partial x}\Psi_2\Big)dx = 2\int_{\mathbb{R}} I(z(x,t))dx. \tag{2.7}$$

3 Multi-symplectic splitting methods for the nonlinear Dirac equation

The Hamiltonian PDE (2.2) can be split into subsystems [26]

$$Mz_t + K_i z_x = \nabla_z S^i(z), \quad i = 1,2,\cdots,N, \tag{3.1}$$

where $K_i^{\mathrm{T}} = -K_i$, $\sum_{i=1}^{N} K_i = K$ and $\sum_{i=1}^{N} S^i(z) = S(z)$. It is easy to know that the subsystems (3.1) have the following multi-symplectic conservation laws

$$\omega_t + (\kappa_i)_x = 0, \tag{3.2}$$

where $\omega = dz/2\wedge Mdz$, $\kappa_i = dz/2\wedge K_i dz$, $i = 1,2,\cdots,N$.

For the NLD equation (1.3), it can be decomposed into one linear subproblem

$$\Psi_t = \mathcal{L}(\Psi) = A\Psi_x + \mathrm{i}mB\Psi \tag{3.3}$$

and one nonlinear subproblem

$$\Psi_t = \mathcal{N}(\Psi) = -2\mathrm{i}\lambda(|\Psi_1|^2 - |\Psi_2|^2)B\Psi, \tag{3.4}$$

where Ψ, A and B are defined as that in (1.1). The above two subproblems can be written as real-value equations, namely

$$\mathcal{L}:\begin{cases} \dfrac{\partial p_1}{\partial t}+\dfrac{\partial p_2}{\partial x}-mq_1=0,\\ \dfrac{\partial q_1}{\partial t}+\dfrac{\partial q_2}{\partial x}+mp_1=0,\\ \dfrac{\partial p_2}{\partial t}+\dfrac{\partial p_1}{\partial x}+mq_2=0,\\ \dfrac{\partial q_2}{\partial t}+\dfrac{\partial q_1}{\partial x}-mp_2=0, \end{cases} \tag{3.5}$$

and

$$\mathcal{N}:\begin{cases} \dfrac{\partial p_1}{\partial t}-2\lambda(p_2^2+q_2^2-p_1^2-q_1^2)q_1=0,\\ \dfrac{\partial q_1}{\partial t}+2\lambda(p_2^2+q_2^2-p_1^2-q_1^2)p_1=0,\\ \dfrac{\partial p_2}{\partial t}+2\lambda(p_2^2+q_2^2-p_1^2-q_1^2)q_2=0,\\ \dfrac{\partial q_2}{\partial t}-2\lambda(p_2^2+q_2^2-p_1^2-q_1^2)p_2=0. \end{cases} \tag{3.6}$$

It is noticed that both the linear subproblem (3.5) and the nonlinear subproblem (3.6) can be written as a multi-symplectic form (3.1) with

$$\begin{aligned} &\mathcal{L}:\ K_1=K,\quad S^1(z)=-\frac{1}{2}m(p_1^2+q_1^2-p_2^2-q_2^2),\\ &\mathcal{N}:\ K_2=0,\quad S^2(z)=\frac{1}{2}\lambda(p_1^2+q_1^2-p_2^2-q_2^2)^2. \end{aligned}$$

In addition, the nonlinear subproblem (3.6) can also be written as an infinite Hamiltonian system

$$z_t=M^{-1}\frac{\delta}{\delta z}H(z),\quad z=[p_1,q_1,p_2,q_2]^{\rm T}, \tag{3.7}$$

with a symplectic conservation law

$$\frac{d}{dt}\int_\Omega \omega dx=0, \tag{3.8}$$

where $\omega=dp_1\wedge dq_1+dp_2\wedge dq_2$ and the Hamiltonian function is

$$H(z)=\frac{\lambda}{2}\int_\Omega (p_2^2+q_2^2-p_1^2-q_1^2)^2dx.$$

3.1 Discretizations for the linear subproblem

We assume that the NLD equation (1.3) is to be integrated in the spacial interval $[x_L,x_R]$ and let $L=x_R-x_L$ be the length of the interval. A uniform grid (x_k,t_n) with space-step $\Delta x=L/N$ and time-step Δt is considered in this paper, where N is the number of spacial subintervals. The space grid points are denoted by $x_k=x_L+(k-1)\Delta x$ and $p_{i,k}$ and $q_{i,k}$ are the approximations to $p_i(x_k,t)$ and $q_i(x_k,t)$, respectively, $i=1,2$, $k=1,2,\cdots,N$. Similarly, $p^n_{i,k}\approx p_i(x_k,t_n)$ and $q^n_{i,k}\approx q_i(x_k,t_n)$.

3.1.1 Multi-symplectic Fourier pseudospectral method for the linear subsystem

In order to apply the Fourier pseudospectral method, we assume that the boundary conditions for the NLD equation (1.3) are periodic, namely $\Psi_i(x_L,t)=\Psi_i(x_R,t)$, $i=1,2$. And let the number of spacial subintervals N to be an even integer in this subsection. In the Fourier pseudospectral method, the first-order partial differential operator ∂_x yields the Fourier spectral differentiation matrix D_1, which is an $N\times N$ skew-symmetric matrix with elements

$$(D_1)_{k,l}=\begin{cases} \dfrac{1}{2}(-1)^{k+l}\mu\cot\Big(\mu\dfrac{x_k-x_l}{2}\Big), & l\neq k,\\ 0, & l=k,\end{cases}\qquad \text{for } k,l=1,2,\cdots,N.$$

Here, $\mu=2\pi/L$. For more details about the Fourier pseudospectral method, see [20] and references therein.

Using the Fourier pseudospectral method in space direction, we can obtain a semi-discretization system for the linear subproblem (3.5)

$$\begin{cases} \dfrac{d\mathbf{p}_1}{dt}+D_1\mathbf{p}_2-m\mathbf{q}_1=0,\\ \dfrac{d\mathbf{q}_1}{dt}+D_1\mathbf{q}_2+m\mathbf{p}_1=0,\\ \dfrac{d\mathbf{p}_2}{dt}+D_1\mathbf{p}_1+m\mathbf{q}_2=0,\\ \dfrac{d\mathbf{q}_2}{dt}+D_1\mathbf{q}_1-m\mathbf{p}_2=0,\end{cases}\tag{3.9}$$

where $\mathbf{p}_i=[p_{i,1},p_{i,2},\cdots,p_{i,N}]^{\mathrm{T}}$ and $\mathbf{q}_i=[q_{i,1},q_{i,2},\cdots,q_{i,N}]^{\mathrm{T}}$, $i=1,2$.

Then, the multi-symplectic Fourier pseudospectral method is obtained while implicit midpoint scheme is implemented in time direction

$$\begin{cases} \mathbf{p}_1^{n+1}=\mathbf{p}_1^n-\Delta t\Big(D_1\mathbf{p}_2^{n+1/2}-m\mathbf{q}_1^{n+1/2}\Big),\\ \mathbf{q}_1^{n+1}=\mathbf{q}_1^n-\Delta t\Big(D_1\mathbf{q}_2^{n+1/2}+m\mathbf{p}_1^{n+1/2}\Big),\\ \mathbf{p}_2^{n+1}=\mathbf{p}_2^n-\Delta t\Big(D_1\mathbf{p}_1^{n+1/2}+m\mathbf{q}_2^{n+1/2}\Big),\\ \mathbf{q}_2^{n+1}=\mathbf{q}_2^n-\Delta t\Big(D_1\mathbf{q}_1^{n+1/2}-m\mathbf{p}_2^{n+1/2}\Big),\end{cases}\tag{3.10}$$

where $\mathbf{p}_i^n=[p_{i,1}^n,p_{i,2}^n,\cdots,p_{i,N}^n]^{\mathrm{T}}$, $\mathbf{p}_i^{n+1/2}=(\mathbf{p}_i^n+\mathbf{p}_i^{n+1})/2$, etc..

Furthermore, Eq. (3.10) can be written as

$$\begin{bmatrix} \mathbf{p}_1^{n+1} \\ \mathbf{q}_1^{n+1} \\ \mathbf{p}_2^{n+1} \\ \mathbf{q}_2^{n+1} \end{bmatrix} = \Big(2\Big(I_{4N}+\frac{1}{2}\Delta tJ_1+\frac{1}{2}\Delta tmJ_2\Big)^{-1}-I_{4N}\Big)\begin{bmatrix} \mathbf{p}_1^{n} \\ \mathbf{q}_1^{n} \\ \mathbf{p}_2^{n} \\ \mathbf{q}_2^{n} \end{bmatrix}, \tag{3.11}$$

where $J_1=\begin{bmatrix} 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & D_1 \\ D_1 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 \end{bmatrix}$, $J_2=\begin{bmatrix} 0 & -I_N & 0 & 0 \\ I_N & 0 & 0 & 0 \\ 0 & 0 & 0 & I_N \\ 0 & 0 & -I_N & 0 \end{bmatrix}$. Here, I_N is the $N\times N$ identity matrix. By direct calculation, we can obtain

$$\Big(I_{4N}+\frac{1}{2}\Delta tJ_1+\frac{1}{2}\Delta tmJ_2\Big)^{-1}=\begin{bmatrix} A & aA & -D_1'B^{-1} & 0 \\ -aA & A & 0 & -D_1'B^{-1} \\ -B^{-1}D_1' & 0 & B^{-1} & -aB^{-1} \\ 0 & -B^{-1}D_1' & aB^{-1} & B^{-1} \end{bmatrix}, \tag{3.12}$$

where $a=\Delta tm/2$, $D_1'=\Delta tD_1/2$, $A=(I+D_1'B^{-1}D_1')/(1+a^2)$, $B=(1+a^2)I-D_1'^2$.

Because B is a symmetric and circulant matrix, we can know its inverse matrix B^{-1} is also a symmetric and circulant matrix [34]. Therefore, in order to obtain B^{-1}, we just need to know its first column. That is to say, we can obtain the matrix $(I_{4N}+\Delta tJ_1/2+\Delta tmJ_2/2)^{-1}$ easily. Therefore, (3.11) is an explicit scheme.

3.1.2 Multi-symplectic Preissmann method for the linear subsystem

The multi-symplectic Fourier pseudospectral method can only be used for periodic boundary conditions. For non-periodic boundary conditions, we have to use other multi-symplectic methods. The multi-symplectic Preissmann scheme is one of the most popular methods which can be used to solve problems with non-periodic boundary conditions.

From [11], we know that the multi-symplectic Preissmann scheme for the linear sub-problem (3.5) can be written as

$$\begin{cases} d_t\mathrm{M}_xp_{1,k}^n+\mathrm{M}_td_xp_{2,k}^n-m\mathrm{M}_t\mathrm{M}_xq_{1,k}^n=0, \\ d_t\mathrm{M}_xq_{1,k}^n+\mathrm{M}_td_xq_{2,k}^n+m\mathrm{M}_t\mathrm{M}_xp_{1,k}^n=0, \\ d_t\mathrm{M}_xp_{2,k}^n+\mathrm{M}_td_xp_{1,k}^n+m\mathrm{M}_t\mathrm{M}_xq_{2,k}^n=0, \\ d_t\mathrm{M}_xq_{2,k}^n+\mathrm{M}_td_xq_{1,k}^n-m\mathrm{M}_t\mathrm{M}_xp_{2,k}^n=0, \end{cases} \tag{3.13}$$

where $d_tp_{1,k}^n=(p_{1,k}^{n+1}-p_{1,k}^n)/\Delta t$, $d_xp_{1,k}^n=(p_{1_{k+1}}^n-p_{1,k}^n)/\Delta x$, $\mathrm{M}_tp_{1,k}^n=(p_{1,k}^{n+1}+p_{1,k}^n)/2$, $\mathrm{M}_xp_{1,k}^n=(p_{1_{k+1}}^n+p_{1,k}^n)/2$, etc..

It is noticed that (3.13) can be written as the compact form

$$\begin{bmatrix} aM & -cM & bD & 0 \\ cM & aM & 0 & bD \\ bD & 0 & aM & cM \\ 0 & bD & -cM & aM \end{bmatrix} \begin{bmatrix} \mathbf{p}_1^{n+1} \\ \mathbf{q}_1^{n+1} \\ \mathbf{p}_2^{n+1} \\ \mathbf{q}_2^{n+1} \end{bmatrix} = \begin{bmatrix} aM & cM & -bD & 0 \\ -cM & aM & 0 & -bD \\ -bD & 0 & aM & -cM \\ 0 & -bD & cM & aM \end{bmatrix} \begin{bmatrix} \mathbf{p}_1^{n} \\ \mathbf{q}_1^{n} \\ \mathbf{p}_2^{n} \\ \mathbf{q}_2^{n} \end{bmatrix}, \tag{3.14}$$

where $a=1/(2\Delta t)$, $b=1/(2\Delta x)$, $c=m/4$,

$$M=\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ 1 & & & 1 \end{bmatrix}_{N\times N}, \qquad D=\begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ 1 & & & -1 \end{bmatrix}_{N\times N}.$$

Furthermore, it can be derived that (3.14) is equivalent to

$$\begin{bmatrix} \mathbf{p}_1^{n+1} \\ \mathbf{q}_1^{n+1} \\ \mathbf{p}_2^{n+1} \\ \mathbf{q}_2^{n+1} \end{bmatrix} = (2aB^{-1}(M\otimes I_4)-I) \begin{bmatrix} \mathbf{p}_1^{n} \\ \mathbf{q}_1^{n} \\ \mathbf{p}_2^{n} \\ \mathbf{q}_2^{n} \end{bmatrix}, \tag{3.15}$$

where

$$B=\begin{bmatrix} aM & -cM & bD & 0 \\ cM & aM & 0 & bD \\ bD & 0 & aM & cM \\ 0 & bD & -cM & aM \end{bmatrix}, \quad B^{-1}=(A^{-1}\otimes I_4)\begin{bmatrix} aM & cM & -bD & 0 \\ -cM & aM & 0 & -bD \\ -bD & 0 & aM & -cM \\ 0 & -bD & cM & aM \end{bmatrix}.$$

Here, $A=(a^2+c^2)M^2-b^2D^2$ and $\otimes$ is the Kronecker inner product.

In fact, since the matrix $2aB^{-1}(M\otimes I_4)-I$ can be expressed as

$$\begin{bmatrix} 2a^2A^{-1}M^2-I & 2acA^{-1}M^2 & -2abA^{-1}MD & 0 \\ -2acA^{-1}M^2 & 2a^2A^{-1}M^2-I & 0 & -2abA^{-1}MD \\ -2abA^{-1}MD & 0 & 2a^2A^{-1}M^2-I & -2acA^{-1}M^2 \\ 0 & -2abA^{-1}MD & 2acA^{-1}M^2 & 2a^2A^{-1}M^2-I \end{bmatrix},$$

we just need to obtain A^{-1} in order to solve the linear equation (3.15) explicitly.

In addition, it is noticed that

$$A=d\begin{bmatrix} 1 & \alpha & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & 1 \\ 1 & & & 1 & \alpha \\ \alpha & 1 & & & 1 \end{bmatrix}=dC(\beta)C(\gamma),$$

where the matrix $C(x)$ denotes $\begin{bmatrix} 1 & x & & \\ & \ddots & \ddots & \\ & & 1 & x \\ x & & & 1 \end{bmatrix}$, $d=a^2-b^2+c^2$, $\alpha=2(a^2+b^2+c^2)/d$, $\beta=(\alpha+\sqrt{\alpha^2-4})/2$ and $\gamma=(\alpha-\sqrt{\alpha^2-4})/2$. And we can derive

$$C^{-1}(x)=\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-4} & a_{N-3} & a_{N-2} & b_{N-1} \\ b_{N-1} & a_0 & a_1 & a_2 & \cdots & a_{N-4} & a_{N-3} & b_{N-2} \\ b_{N-2} & b_{N-1} & a_0 & a_1 & a_2 & \cdots & a_{N-4} & b_{N-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ b_4 & \cdots & b_{N-2} & b_{N-1} & a_0 & a_1 & a_2 & b_3 \\ b_3 & b_4 & \cdots & b_{N-2} & b_{N-1} & a_0 & a_1 & b_2 \\ b_2 & b_3 & b_4 & \cdots & b_{N-2} & b_{N-1} & a_0 & b_1 \\ b_1 & b_2 & b_3 & b_4 & \cdots & b_{N-2} & b_{N-1} & b_0 \end{bmatrix},$$

where $a_k=(-x)^k\xi$, $b_k=(-x)^k/\eta$ for $k=0,1,2,\cdots,N-1$, and $\xi=1+(-x)^N/\eta$, $\eta=1-(-x)^N$. Hence the inverse matrix of A can be obtained exactly. Therefore, the scheme (3.15) is explicit.

3.2 Discretization for the nonlinear subproblem

For the nonlinear subproblem (3.6), the point-wise accuracy solution can be computed as [35]

$$\begin{cases} \Psi_{1,k}^{n+1}=\Psi_{1,k}^{n}e^{-2\mathrm{i}\lambda\Delta t(|\Psi_{2,k}^n|^2-|\Psi_{1,k}^n|^2)}, \\ \Psi_{2,k}^{n+1}=\Psi_{2,k}^{n}e^{2\mathrm{i}\lambda\Delta t(|\Psi_{2,k}^n|^2-|\Psi_{1,k}^n|^2)}. \end{cases} \tag{3.16}$$

Namely

$$\begin{cases} \begin{bmatrix} p_{1,k}^{n+1} \\ q_{1,k}^{n+1} \end{bmatrix} = \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{bmatrix} \begin{bmatrix} p_{1,k}^{n} \\ q_{1,k}^{n} \end{bmatrix}, \\ \begin{bmatrix} p_{2,k}^{n+1} \\ q_{2,k}^{n+1} \end{bmatrix} = \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix} \begin{bmatrix} p_{2,k}^{n} \\ q_{2,k}^{n} \end{bmatrix}, \end{cases} \tag{3.17}$$

where $\theta_k=2\lambda\Delta t(|\Psi_{2,k}^n|^2-|\Psi_{1,k}^n|^2)$.

It is easy to derive from (3.17) that $dp_{1,k}^{n+1}\wedge dq_{1,k}^{n+1}=dp_{1,k}^{n}\wedge dq_{1,k}^{n}$ and $dp_{2,k}^{n+1}\wedge dq_{2,k}^{n+1}=dp_{2,k}^{n}\wedge dq_{2,k}^{n}$. That is to say, the scheme (3.17) preserves the total discrete symplecticity $\Sigma_k\omega_k^n\Delta x$ of the nonlinear subproblem exactly, where $\omega_k^n=dp_{1,k}^n\wedge dq_{1,k}^n+dp_{2,k}^n\wedge dq_{2,k}^n$. Hence, (3.17) is a symplectic scheme.

In this paper, we choose the second order Strang splitting method

$$\Psi_i(x,t+\Delta t)=\exp\Big(\frac{\Delta t}{2}\mathcal{N}\Big)\exp(\Delta t\mathcal{L})\exp\Big(\frac{\Delta t}{2}\mathcal{N}\Big)\Psi_i(x,t),\quad i=1,2, \tag{3.18}$$

to combine the solutions of the linear subproblem and the nonlinear subproblem to obtain the final schemes for the NLD equation (1.3). If the mult-symplectic Fourier pseudospectral method is used to solve the linear problem in (3.18), the method is referred to as multi-symplectic splitting Fourier pseudospectral method (MSSFPM). On the other hand, if multi-symplectic Preissmann scheme is used, the method is referred to as multi-symplectic splitting Preissmann method (MSSPM). As the linear subproblem and the nonlinear subproblem can be both integrated explicitly, we state that the MSSFPM and the MSSPM are both explicit methods. Since multi-symplectic and symplectic methods are employed to integrate the linear subsystem and nonlinear subsystem, respectively, the proposed methods have a good long time stable property in each step.

Furthermore, under periodic or vanishing boundary conditions, the same total discrete symplecticity $\Sigma_k \omega_k^n \Delta x$ as the nonlinear subproblem is preserved by the multi-symplectic methods which are used to solve the linear subproblem. So the proposed splitting methods in this paper can preserve the total symplecticity of the NLD equation in such boundary conditions.

4 Numerical experiments

In this section, we choose periodic boundary condition $\Psi(x_L,t)=\Psi(x_R,t)$ to show the effectiveness of our proposed methods, where x_L and x_R are real constants. And according to (1.2), (2.6) and (2.7), we define the errors in discrete charge, energy and momentum as

$$\sum_{k=1}^{N}(|\Psi_{1,k}^n|^2+|\Psi_{2,k}^n|^2-|\Psi_{1,k}^0|^2-|\Psi_{2,k}^0|^2)\Delta x, \tag{4.1a}$$

$$-2\sum_{k=1}^{N}(E_k^n-E_k^0)\Delta x, \tag{4.1b}$$

and

$$2\sum_{k=1}^{N}(I_k^n-I_k^0)\Delta x, \tag{4.1c}$$

respectively, where $E_k^n \approx E(z_k^n)$ and $I_k^n \approx I(z_k^n)$. For the MSSFPM, we use $(D_1\mathbf{p}_i^n)_k$ and $(D_1\mathbf{q}_i^n)_k$ to approximate $\partial/\partial x p_{i,k}^n$ and $\partial/\partial x q_{i,k}^n$ that appear in E_k^n and I_k^n, respectively, $i=1,2$. For the MSSPM, we choose $d_x p_{i,k}^n$ and $d_x q_{i,k}^n$ as the approximations.

Example 4.1. When the constants $m=1$ and $\lambda=1/2$ in (1.3), the NLD equation has the following theoretical solitary wave solution [1]

$$\Psi^{sw}(x,t)=[\Psi_1^{sw},\Psi_2^{sw}]^{\mathrm{T}}=[M(x),\mathrm{i}N(x)]^{\mathrm{T}}e^{-\mathrm{i}\Lambda t}, \tag{4.2}$$

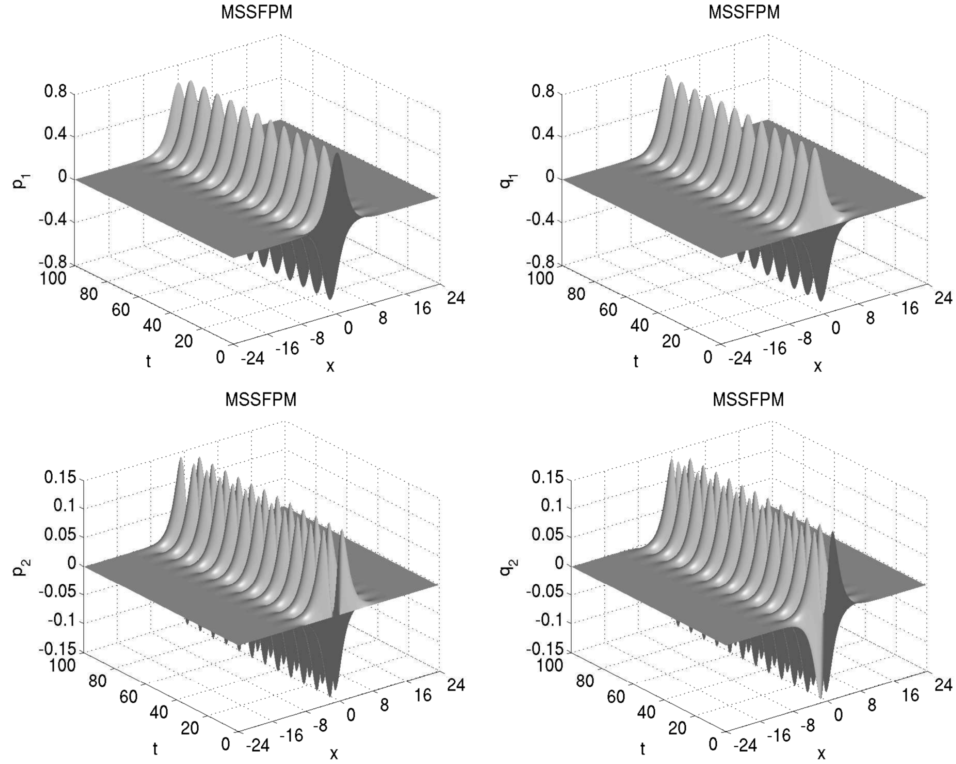

Figure 1: The four numerical solitary wave functions obtained by using MSSFPM with initial condition $\Psi^{sw}(x,0)$, $\Lambda=0.75$, $[x_L,x_R]=[-24,24]$, $N=160$, $\Delta t=0.01$.

where

$$M(x)=(2(1-\Lambda^2))^{1/2}(1+\Lambda)^{1/2}\frac{\cosh((1-\Lambda^2)^{1/2}x)}{1+\Lambda\cosh(2(1-\Lambda^2)^{1/2}x)}, \tag{4.3a}$$

$$N(x)=(2(1-\Lambda^2))^{1/2}(1-\Lambda)^{1/2}\frac{\sinh((1-\Lambda^2)^{1/2}x)}{1+\Lambda\cosh(2(1-\Lambda^2)^{1/2}x)}, \tag{4.3b}$$

and the frequency Λ is a real constant.

The numerical results for the solitary wave solution (4.2) with $\Lambda=0.75$ are showed in Fig. 1 and Fig. 2.

From these two figures, we can see that both the MSSFPM and the MSSPM perform well. The time interval $[0,100]$ contains almost 12 periods, which matches with the theoretical solution. The corresponding errors in the charge, energy and momentum for the two proposed methods are shown in Fig. 3.

From this figure, we can see that both of the methods preserve the charge, energy and momentum well. Compared with the MSSPM, nevertheless, the MSSFPM performs better in preserving the global conservation laws.

In addition, a numerical accuracy test of spatial direction is also presented in Table 1.

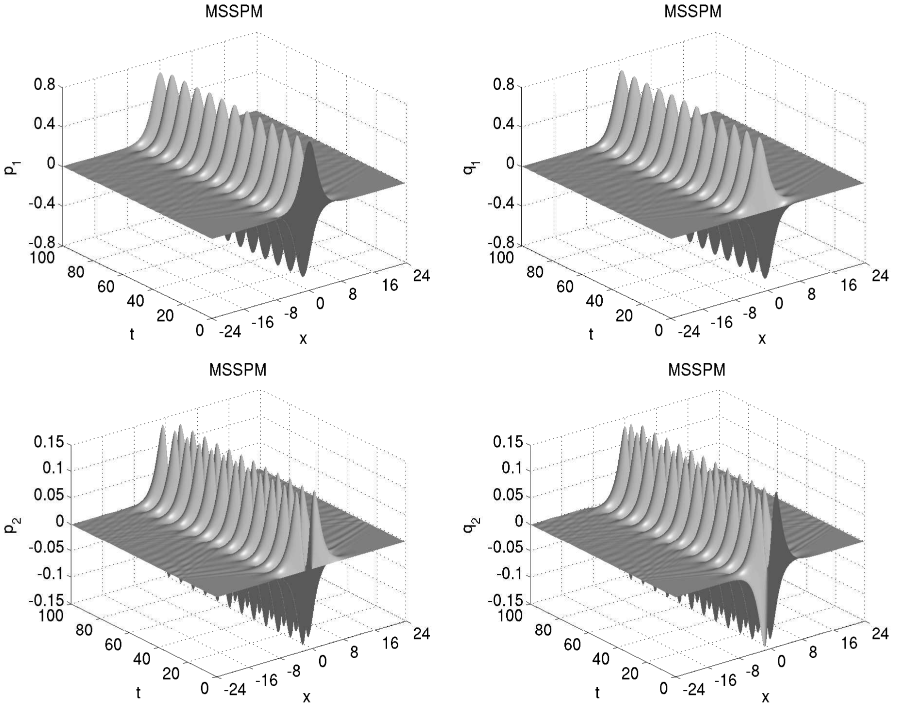

Figure 2: The four numerical solitary wave functions obtained by using MSSPM with initial condition $\Psi^{sw}(x,0)$, $\Lambda=0.75$, $[x_L,x_R]=[-24,24]$, $N=160$, $\Delta t=0.01$.

For the MSSFPM, we observe that the errors decrease until $N=160$. When the MSSFPM achieves its maximum efficiency, a larger number of nodes does not improve the accuracy of the results. This phenomenon is due to the non-periodicity of the real problem. To further improve the result, we should use larger interval $[x_L,x_R]$ to do the simulation. For the MSSPM, we can see that the scheme is of approximately second order accuracy in space direction. In such a situation, $N=20$ is not large enough to obtain satisfied results due to the lower order accuracy in space direction compared with the MSSFPM.

Example 4.2. When $m=1$ and $\lambda=1/2$, there is another exact solution of the NLD equation (1.3) which represents a solitary wave traveling with velocity v [10], that solution is

$$\Psi^{ss}=[\Psi_1^{ss}(x,t),\Psi_2^{ss}(x,t)]^{\rm T}, \tag{4.4}$$

where

$$\Psi_1^{ss}(x,t)=\sqrt{\frac{\gamma+1}{2}}\Psi_1^{sw}(\tilde{x},\tilde{t})+\mathrm{sign}(v)\sqrt{\frac{\gamma-1}{2}}\Psi_2^{sw}(\tilde{x},\tilde{t}), \tag{4.5a}$$

$$\Psi_2^{ss}(x,t)=\sqrt{\frac{\gamma+1}{2}}\Psi_2^{sw}(\tilde{x},\tilde{t})+\mathrm{sign}(v)\sqrt{\frac{\gamma-1}{2}}\Psi_1^{sw}(\tilde{x},\tilde{t}). \tag{4.5b}$$

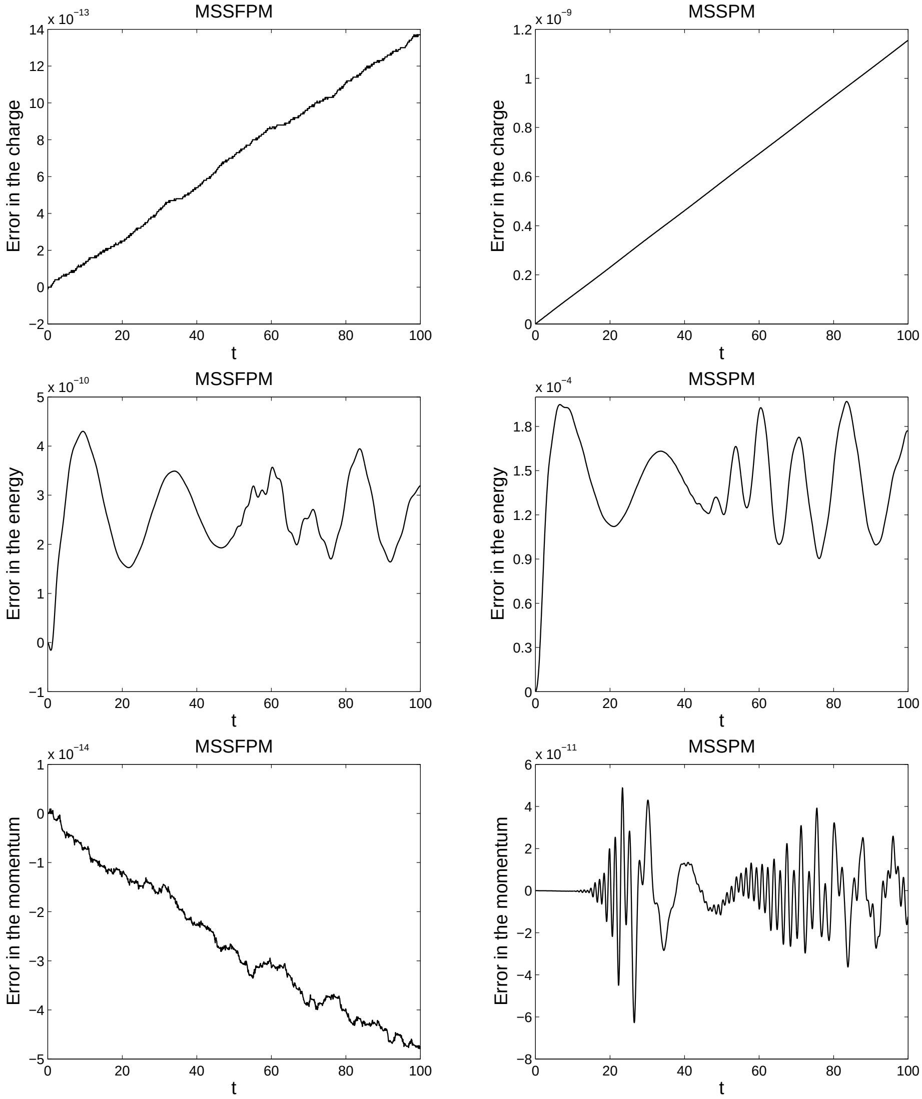

Figure 3: Corresponding errors in the discrete charge, energy and momentum obtained by the MSSFPM and the MSSPM.

Here, $\gamma=1/\sqrt{1-v^2}$, $\tilde{x}=\gamma(x-vt)$, $\tilde{t}=\gamma(t-vx)$, Ψ_1^{sw} and Ψ_2^{sw} are defined in (4.2) and $\mathrm{sign}(v)$ denotes the sign of v. When $v>0$, the wave travels from left to right and, when $v<0$, from right to left; when $v=0$, it does not move and we get the standing wave (4.2). Because the MSSFPM and the MSSPM give similar numerical results, we just present the results obtained by the MSSFPM in this subsection. In addition, the charge density of the NLD equation (1.3) is defined by

$$\rho_Q(x,t)=|\Psi_1|^2+|\Psi_2|^2. \tag{4.6}$$

Table 1: The accuracy test of the NLD equation (1.3) with the solution (4.2) in the space direction at time $t=1$, $\Delta t=0.0001$, $[x_L,x_R]=[-24,24]$.

	N	MSSFPM		MSSPM			
		L^2 error	L^∞ error	L^2 error	order	L^∞ error	order
p_1	20	7.08E-02	3.02E-02				
	40	1.24E-02	6.16E-03	5.00E-02	-	3.35E-02	-
	80	1.04E-05	8.34E-06	1.07E-02	2.22	8.21E-03	2.03
	160	6.62E-06	3.61E-06	2.53E-03	2.08	1.98E-03	2.05
	320	6.62E-06	3.61E-06	6.37E-04	1.99	5.14E-04	1.95
q_1	20	1.16E-01	6.39E-02				
	40	8.41E-03	3.22E-03	1.81E-02	-	7.05E-03	-
	80	1.29E-05	8.10E-06	3.87E-03	2.23	2.72E-03	1.37
	160	7.11E-06	3.88E-06	1.01E-03	1.94	8.49E-04	1.68
	320	7.11E-06	3.88E-06	2.84E-04	1.83	2.49E-04	1.77
p_2	20	8.67E-02	2.34E-02				
	40	2.28E-03	6.21E-04	2.43E-02	-	1.31E-02	-
	80	1.22E-05	4.46E-06	5.61E-03	2.11	3.45E-03	1.92
	160	1.62E-06	7.90E-07	1.39E-03	2.01	8.53E-04	2.02
	320	1.62E-06	7.91E-07	3.57E-04	1.96	2.21E-04	1.95
q_2	20	7.13E-02	1.92E-02				
	40	1.09E-02	3.30E-03	3.34E-02	-	1.91E-02	-
	80	4.39E-06	2.35E-06	7.57E-03	1.39	5.35E-03	1.84
	160	1.51E-06	7.36E-07	1.73E-03	2.13	1.28E-03	2.06
	320	1.51E-06	7.37E-07	4.27E-04	2.02	3.28E-04	1.96

4.1 Propagation of one soliton

We use the initial value

$$\Psi(x,0)=\Psi^{ss}(x-x_0,0) \tag{4.7}$$

to simulate one soliton of the NLD equation (1.3). The propagation of one soliton is shown in Fig. 4. The corresponding global errors are also presented in this figure. From the figure, we can see that the solitons transmits without any change with respect to shapes and velocities; the charge, energy and momentum are preserved very well.

4.2 Collision of double solitons

The collision of two solitons of the NLD equation (1.3) is presented in Fig. 5 with initial condition

$$\Psi(x,0)=\Psi_l^{ss}(x-x_l,0)+\Psi_r^{ss}(x-x_r,0). \tag{4.8}$$

From this figure, we can see the interaction of these solitons is elastic when $\Lambda=0.5$. Fluctuations in the errors of energy and momentum are clearly observed at about $t=40$ when collision happened in this case. While $\Lambda=0.2$, the interaction of two solitons is inelastic, oscillatory waves are generated when two solitons collided. In this case, however, the symmetry of the solution is preserved very well.

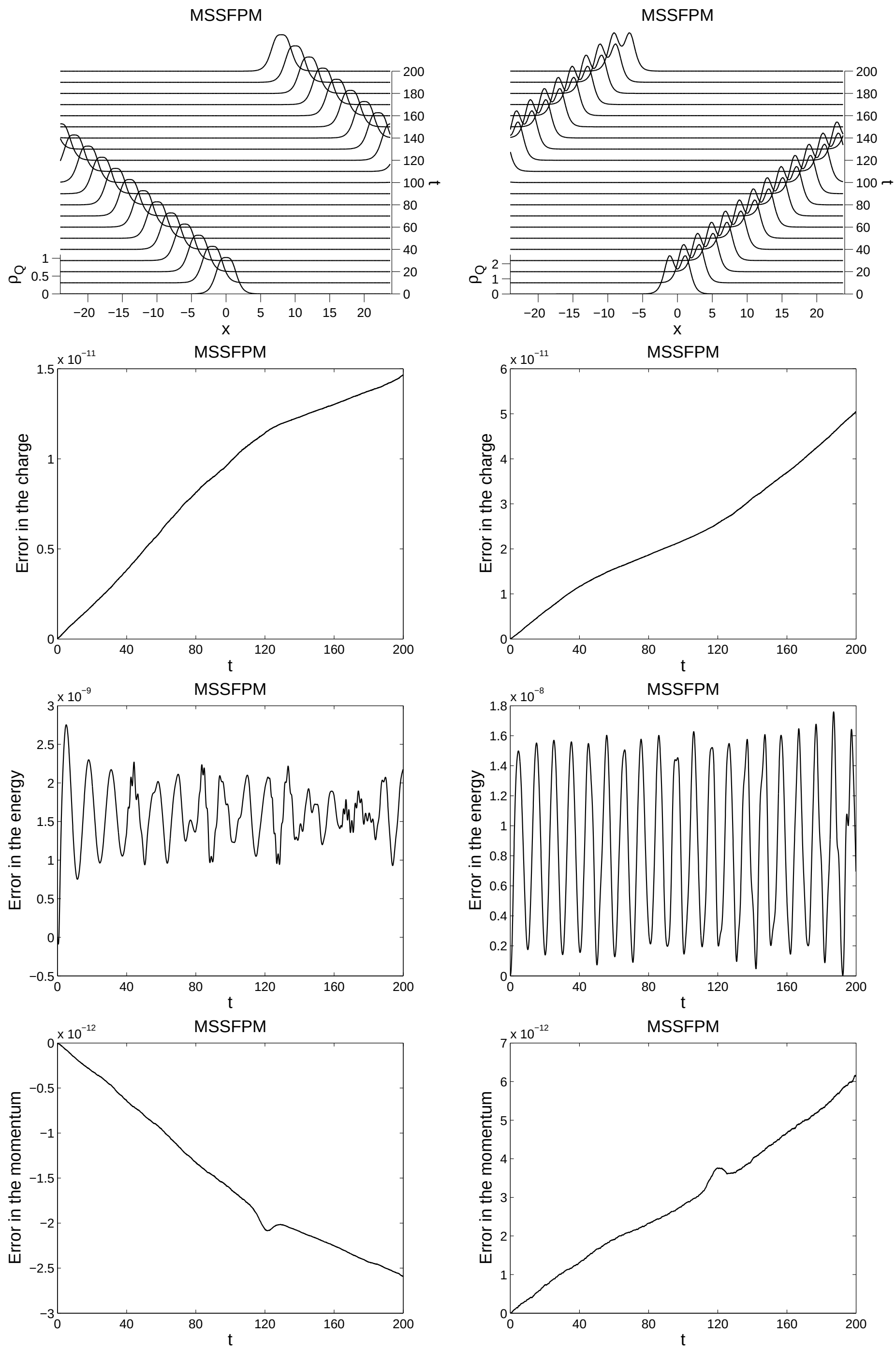

Figure 4: Propagation of one soliton of the NLD equation (1.3) and its corresponding errors obtained by MSSFPM with $x_0=0$, $[x_L,x_R]=[-24,24]$, $N=200$, $\Delta t=0.01$. Left: $\Lambda=0.5$, $v=-0.2$, right: $\Lambda=0.2$, $v=0.2$.

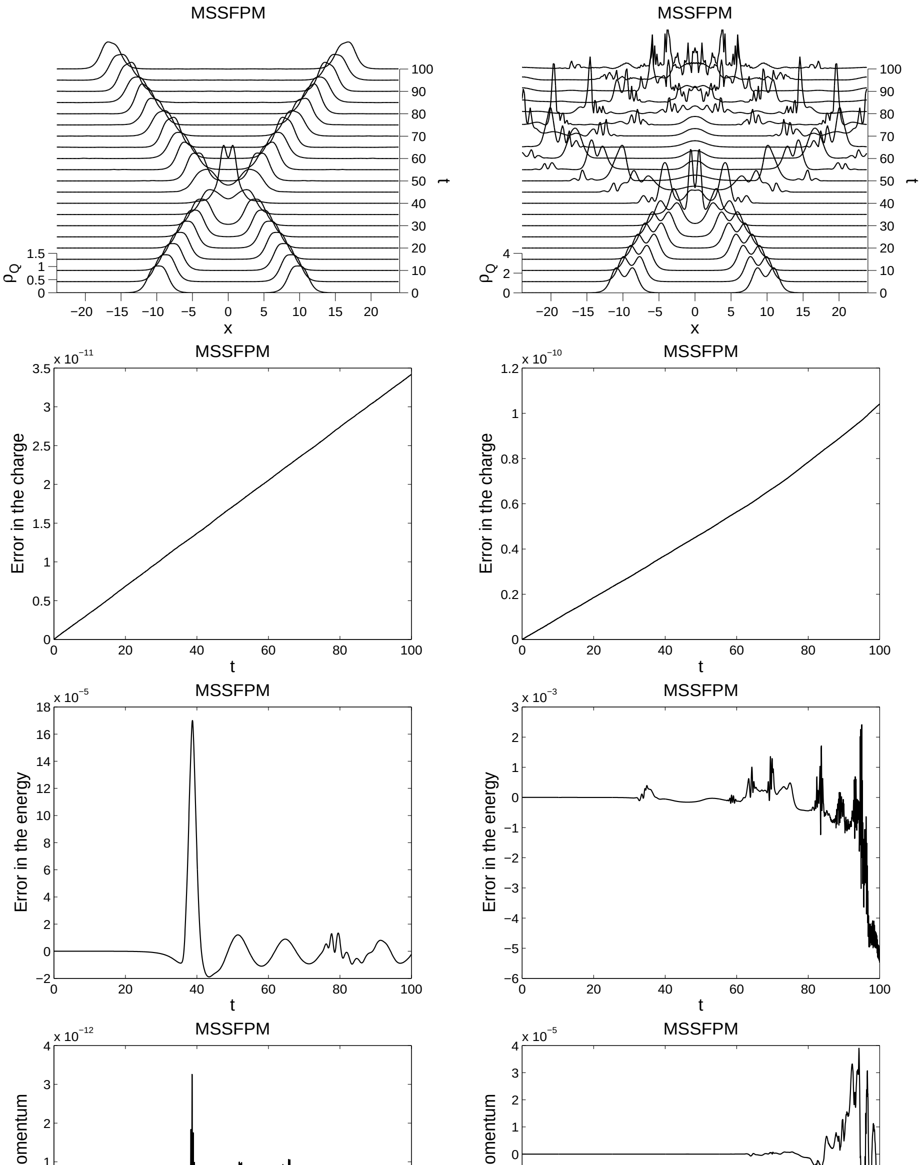

Figure 5: Collision of two solitons of the NLD equation (1.3) and their corresponding errors obtained by MSSFPM with $v_l=-v_r=0.2$ and $x_r=-x_l=10$, $[x_L,x_R]=[-24,24]$, $N=300$, $\Delta t=0.01$. Left: $\Lambda_l=\Lambda_r=0.5$, right: $\Lambda_l=\Lambda_r=0.2$.

4.3 Interaction of triple solitons

The NLD equation (1.3) admits three solitons under the initial condition

$$\Psi(x,0)=\Psi_l^{ss}(x-x_l,0)+\Psi_m^{ss}(x-x_m,0)+\Psi_r^{ss}(x-x_r,0). \tag{4.9}$$

The interaction of triple solitons are shown in Fig. 6. The corresponding errors of (4.1a), (4.1b) and (4.1c) are also illustrated in this figure. We can see that the collision is almost elastic. After collision, the amplitudes of the left wave and the right wave become larger while the amplitude of the middle wave becomes smaller. From the corresponding errors we can conclude that the charge, energy and momentum are preserved well. While collision happened, fluctuation is observed in the error of the energy.

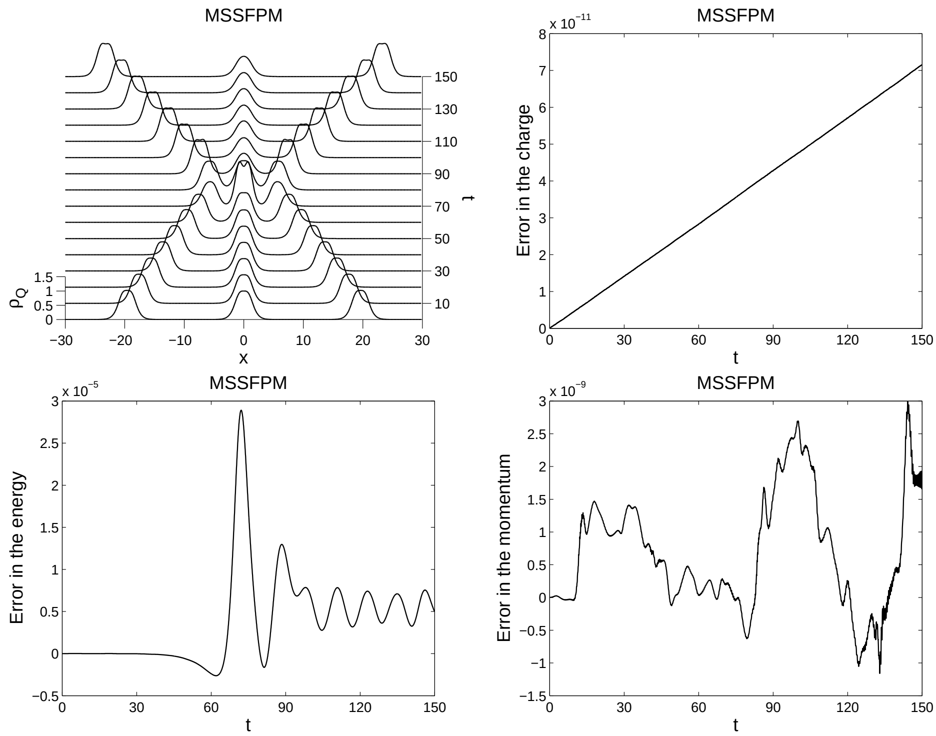

Figure 6: Interaction of three solitons of the NLD equation (1.3) and their corresponding errors obtained by MSSFPM with $v_l=-v_r=0.2$, $v_m=0.0$, $x_r=-x_l=20$, $x_m=0$, $\Lambda_l=\Lambda_m=\Lambda_r=0.5$, $[x_L,x_R]=[-30,30]$, $N=300$, $\Delta t=0.01$.

5 Conclusions

In this paper, we propose two multi-symplectic splitting methods to solve the nonlinear Dirac equation. We find that these two methods can be solved explicitly and preserve the total symplecticity of the NLD equation under periodic or vanishing boundary conditions. From the results of numerical experiments, we can see both the MSSFPM and the MSSPM are effective. The MSSFPM, nevertheless, has better performance in preserving the discrete charge, energy and momentum under the same spatial grids due to its high accuracy in space direction. The numerical results also show that the proposed methods are effective in simulating interaction of soliton solutions.

Acknowledgments

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