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An Inverse Problem of Determining Coefficients in a One-Dimensional Radiative Transport Equation

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Abstract. We consider an inverse problem of determining unknown coefficients for a one-dimensional analogue of radiative transport equation. We show that some combination of the unknown coefficients can be uniquely determined by giving pulse-like inputs at the boundary and observing the corresponding outputs. Our result can be applied for determination of absorption and scattering properties of an optically turbid medium if the radiative transport equation is appropriate for describing the propagation of light in the medium.

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Key words: Inverse problem, radiative transport equation, first-order hyperbolic system, optical tomography.

1 Introduction

We consider a one-dimensional version of the inverse problem of identifying unknown coefficients of the one-speed, time-dependent radiative transport equation. This inverse problem is related with the study of optical tomography (see, e.g., [1, 2] and references therein). Optical tomography has been studied for several decades as a new modality of medical imaging technique using low-energy light in the near-infrared region. Compared with other tomographic techniques using high-energy radiation (e.g., X-ray CT), optical tomography is considered to be less harmful to human body. In most researches on optical tomography the propagation of near-infrared light in biological tissues is modeled by the radiative transport equation, and the process of imaging is formulated as an inverse problem of determining unknown coefficients of the equation. Although the original problem is three-dimensional in space variables, our discussion here is limited to the

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one-dimensional case partly because the three-dimensional problem is quite difficult and mainly because we can obtain a reconstruction formula for the unknown coefficients.

Let $(0,H)=\{x\in\mathbb{R};\, 0<x<H\}$ be a finite open interval in $\mathbb{R}$. Let $\mu_{\rm a}(x)$, $\mu_{\rm s}(x)$, and $q(x)$ be continuous functions on the closed interval $[0,H]$ and assume that $\mu_{\rm a}(x)\ge 0$, $\mu_{\rm s}(x)\ge 0$, and $0\le q(x)\le 1$ there. We consider the situation where the interval $[0,H]$ is occupied by a medium that absorbs and scatters photons, with $\mu_{\rm a}$, $\mu_{\rm s}$, and q being the distribution of the absorption coefficient, the scattering coefficient, and the probability of backward scattering, respectively. Let $I_1(x,t)$ be the density of photons moving through the medium with speed c in the positive x direction, and $I_2(x,t)$ in the negative x direction. The time evolution of I_1 and I_2 are described by the system of differential equations

$$\frac{1}{c}\frac{\partial I_1}{\partial t}+\frac{\partial I_1}{\partial x}=-(\mu_{\rm a}+q\mu_{\rm s})I_1+q\mu_{\rm s}I_2, \quad 0<x<H, \quad 0<t<T, \tag{1.1a}$$

$$\frac{1}{c}\frac{\partial I_2}{\partial t}-\frac{\partial I_2}{\partial x}=-(\mu_{\rm a}+q\mu_{\rm s})I_2+q\mu_{\rm s}I_1, \quad 0<x<H, \quad 0<t<T, \tag{1.1b}$$

where c and T are positive numbers. We always assign the initial condition

$$I_1(x,0)=I_2(x,0)=0, \quad 0\le x\le H. \tag{1.2}$$

We assume that the speed c is a known constant, while the coefficients $\mu_{\rm a}(x)$, $\mu_{\rm s}(x)$, and $q(x)$ are unknown functions. In order to determine those unknowns, we consider an experiment as follows. We give a pulse-like input at one end of the interval $[0,H]$ and observe the boundary values of the outward flow at both ends, i.e., $I_1(H,t)$ and $I_2(0,t)$. We again follow the same process by giving the input at the other end. To be precise, we solve (1.1) and (1.2) with the boundary condition

$$I_1(0,t)=\delta(t), \quad I_2(H,t)=0. \tag{1.3}$$

Writing the solution to (1.1), (1.2), and (1.3) as $I^1=(I_1^1,I_2^1)$, we observe

$$I_1^1(H,t), \quad I_2^1(0,t), \quad 0\le t\le T. \tag{1.4}$$

Next we solve (1.1) and (1.2) with the boundary condition

$$I_1(0,t)=0, \quad I_2(H,t)=\delta(t), \tag{1.5}$$

write the solution to (1.1), (1.2), and (1.5) as $I^2=(I_1^2,I_2^2)$, and then observe

$$I_1^2(H,t), \quad I_2^2(0,t), \quad 0\le t\le T. \tag{1.6}$$

Our main result is as follows.

Theorem 1.1. *Let m and M be positive numbers with $m<M$. In the setting above, we consider the admissible set $A(m,M)$ of the unknown coefficients satisfying*

$$\mu_a,\ \mu_s,\ q\in C^0[0,H] \quad \text{and} \quad \mu_a,\mu_s\ge 0, \quad 0\le q\le 1, \quad m\le \mu_a+q\mu_s\le M \ \text{ on } \ [0,H].$$

Then there exists a number $H^*>0$, such that if $0<H<H^*$, $T\ge 2H/c$, and $(\mu_a,\mu_s,q)\in A(m,M)$, the data (1.4) *and* (1.6) *uniquely identify μ_a and $q\mu_s$.*

We remark that the system (1.1) depends on $\mu_{\rm s}$ and q only through the product $q\mu_{\rm s}$. In other words, we cannot reconstruct $\mu_{\rm s}$ and q separately.

We prove Theorem 1.1 by a method explained in [4, Chap. 5] that is applicable to the inverse coefficient problems for general first-order hyperbolic systems. Hence the present paper is in fact an application of it to a simple case: the system (1.1) is a symmetric hyperbolic system with only two components and we actually reconstruct two functions $\mu_{\rm a}$ and $q\mu_{\rm s}$. The simplicity of the setting, however, leads to obtaining reconstruction formulae (3.4) and (3.5) for them.

In experimental study of optical tomography, it is an important step to determine the absorption and scattering property of a phantom made from an optically turbid material. To our knowledge a decisive method for doing so is still unknown, and we expect that the theorem above is applicable for that purpose.

2 Solution to the direct problem

From now on we put $D:=(0,H)\times(0,T)=\{(x,t)\in\mathbb{R}^2|0<x<H,\ 0<t<T\}$. Here we investigate the solution to the direct problem for the system

$$\left[\frac{\partial}{\partial t}+c\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{\partial}{\partial x}+c\begin{pmatrix}\mu_{\rm a}+q\mu_{\rm s}&-q\mu_{\rm s}\\-q\mu_{\rm s}&\mu_{\rm a}+q\mu_{\rm s}\end{pmatrix}\right]\begin{pmatrix}I_1^k\\I_2^k\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\quad (x,t)\in D, \tag{2.1}$$

with initial-boundary conditions

$$I_1^k(x,0)=I_2^k(x,0)=0,\qquad 0\le x\le H, \tag{2.2a}$$

$$I_1^k(0,t)=\delta(t)\delta_{1k},\qquad I_2^k(H,t)=\delta(t)\delta_{2k}, \tag{2.2b}$$

for $k\in\{1,2\}$, where $\delta(t)$ is the Dirac delta function and δ_{ik} is the Kronecker delta. We introduce new dependent variables $\bar{I}^k=(\bar{I}_1^k,\bar{I}_2^k)$ by

$$\bar{I}_i^k(x,t):=p_i(x)I_i^k(x,t),\qquad i,k\in\{1,2\}, \tag{2.3}$$

where

$$p_1(x)=\exp\Big(\int_0^x(\mu_{\rm a}+q\mu_{\rm s})(\xi)d\xi\Big), \tag{2.4a}$$

$$p_2(x)=\exp\Big(-\int_0^x(\mu_{\rm a}+q\mu_{\rm s})(\xi)d\xi\Big)=\frac{1}{p_1(x)}. \tag{2.4b}$$

Then the direct problem (2.1)-(2.2) is equivalent to

$$\left[\frac{\partial}{\partial t}+\begin{pmatrix}c&0\\0&-c\end{pmatrix}\frac{\partial}{\partial x}+\begin{pmatrix}0&a_{12}\\a_{21}&0\end{pmatrix}\right]\begin{pmatrix}\bar{I}_1^k\\\bar{I}_2^k\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\quad (x,t)\in D, \tag{2.5a}$$

$$\bar{I}_1^k(x,0)=\bar{I}_2^k(x,0)=0,\quad 0\le x\le H, \tag{2.5b}$$

$$\bar{I}_1^k(0,t)=\delta_{1k}\delta(t),\quad \bar{I}_2^k(H,t)=\delta_{2k}p_2(H)\delta(t), \tag{2.5c}$$

where the new coefficients $a_{12}=a_{12}(x)$ and $a_{21}=a_{21}(x)$ are given by

$$a_{12}:=-\frac{cq\mu_s p_1}{p_2}=-cq\mu_s p_1^2,\quad a_{21}:=-\frac{cq\mu_s p_2}{p_1}=-cq\mu_s p_1^{-2}. \tag{2.6}$$

The next lemma shows the unique existence and the structure of the solution to the direct problem (2.5).

Lemma 2.1. *If the coefficients are continuous on the closed interval* $[0,H]$, *then for each* $k\in\{1,2\}$ *the problem* (2.5) *is uniquely solvable, and the solution is written in the form*

$$\bar{I}_1^k(x,t)=\delta_{1k}\delta\Big(t-\frac{x}{c}\Big)+w_1^k(x,t), \tag{2.7a}$$

$$\bar{I}_2^k(x,t)=p_2(H)\delta_{2k}\delta\Big(t-\frac{H-x}{c}\Big)+w_2^k(x,t), \tag{2.7b}$$

where $w^1=(w_1^1,w_2^1)$ *is the unique solution to the system of integral equations*

$$w_1^1(x,t)+\int_{t_1(x,t)}^t a_{12}(\xi)w_2^1(\xi,\tau)\Big|_{\xi=c\tau+x-ct}d\tau=0, \tag{2.8a}$$

$$w_2^1(x,t)+\int_{t_2(x,t)}^t a_{21}(\xi)w_1^1(\xi,\tau)\Big|_{\xi=-c\tau+x+ct}d\tau=-F_2^1(x,t), \tag{2.8b}$$

and $w^2=(w_1^2,w_2^2)$ *is the unique solution to*

$$w_1^2(x,t)+\int_{t_1(x,t)}^t a_{12}(\xi)w_2^2(\xi,\tau)\Big|_{\xi=c\tau+x-ct}d\tau=-F_1^2(x,t), \tag{2.9a}$$

$$w_2^2(x,t)+\int_{t_2(x,t)}^t a_{21}(\xi)w_1^2(\xi,\tau)\Big|_{\xi=-c\tau+x+ct}d\tau=0, \tag{2.9b}$$

with the following notation:

$$\begin{aligned}
&t_1(x,t):=\max\Big\{0,t-\frac{x}{c}\Big\},\qquad t_2(x,t):=\max\Big\{0,t-\frac{H-x}{c}\Big\},\\
&F_2^1(x,t):=\frac{1}{2}a_{21}\Big(\frac{x+ct}{2}\Big)\times\begin{cases}1, & (|ct-H|\le H-x),\\ 0, & (|ct-H|>H-x),\end{cases}\\
&F_1^2(x,t):=\frac{p_2(H)}{2}a_{12}\Big(\frac{x-ct+H}{2}\Big)\times\begin{cases}1, & (|ct-H|\le x),\\ 0, & (|ct-H|>x).\end{cases}
\end{aligned}$$

On the closure $\overline{D}=[0,H]\times[0,T]$ *of* D, *the function* w_i^1 *(resp.* w_i^2*) is bounded and piecewise continuous with jump discontinuity along the lines of discontinuity of the function* F_2^1 *(resp.* F_1^2*).*

Proof. We only give an outline. Inserting the expression (2.7) into (2.5) yields the following system for $w^k=(w_1^k,w_2^k)$:

$$\Big(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\Big)w_1^k+a_{12}w_2^k+a_{12}p_2(H)\delta_{2k}\delta\Big(t-\frac{H-x}{c}\Big)=0, \tag{2.10a}$$

$$\Big(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\Big)w_2^k+a_{21}w_1^k+a_{21}\delta_{1k}\delta\Big(t-\frac{x}{c}\Big)=0, \tag{2.10b}$$

$$w_1^k(x,0)=w_2^k(x,0)=0, \tag{2.10c}$$

$$w_1^k(0,t)=0, \qquad w_2^k(H,t)=0. \tag{2.10d}$$

For $k=1$ we obtain the system (2.8) by integrating the Eqs. (2.10a) and (2.10b) from the boundary ∂D to the point (x,t) with the help of (2.10c) and (2.10d). By putting

$$\begin{aligned}
U(x,t)&:=\int_{t_1(x,t)}^t a_{12}(\xi)F_2^1(\xi,\tau)\Big|_{\xi=c\tau+x-ct}d\tau,\\
V(x,t)&:=\int_{t_2(x,t)}^t a_{21}(\xi)U(\xi,\tau)\Big|_{\xi=-c\tau+x+ct}d\tau,\\
z_1&:=w_1^1-U, \qquad z_2:=w_2^1+F_2^1,
\end{aligned}$$

the system (2.8) is equivalent to

$$z_1(x,t)+\int_{t_1(x,t)}^t a_{12}(\xi)z_2(\xi,\tau)\Big|_{\xi=c\tau+x-ct}d\tau=0, \tag{2.11a}$$

$$z_2(x,t)+\int_{t_2(x,t)}^t a_{21}(\xi)z_1(\xi,\tau)\Big|_{\xi=-c\tau+x+ct}d\tau=-V(x,t). \tag{2.11b}$$

The function $U(x,t)$ is defined by integrating $a_{12}(\xi)F_2^1(\xi,\tau)$ up to the point (x,t) along the line $L:\xi-c\tau=x-ct$, and the integrand is bounded and discontinuous along the lines $\xi-c\tau=0$ (parallel to L) and $\xi+c\tau=2H$ (transversal). Therefore $U(x,t)$ is discontinuous only along the line $x-ct=0$. Analogous reasoning shows that the function $V(x,t)$ is continuous on $\overline{D}$. Then the system (2.11) can be seen as a linear equation

$$(I_Z+A)\begin{pmatrix}z_1\\z_2\end{pmatrix}=\begin{pmatrix}0\\-V\end{pmatrix}$$

in the Banach space $Z:=C^0(\overline{D})\times C^0(\overline{D})$ with the maximum norm, where I_Z is the identity on Z and the operator A denotes the integral terms of (2.11). The spectral radius of A is seen to be zero in a way similar to the linear Volterra integral operator on a compact interval with continuous kernel (see, e.g., [3, Chap. 3]). Hence the operator I_Z+A has a bounded inverse and the Eq. (2.11) is uniquely solvable in Z.

Similarly, for $k=2$ we obtain the system (2.9) and its unique solvability. □

3 Outline of a proof of Theorem 1.1

We begin by a remark on the functions w_i^k and by setting of a function space. We see from (2.8) and (2.9) that

$$\begin{aligned} &w_i^1(x,t)=0 \quad \text{for } ct<x,\\ &w_i^2(x,t)=0 \quad \text{for } x<H-ct. \end{aligned}$$

To prove Theorem 1.1 we aim at reconstructing the functions w_i^k, a_{12}, and a_{21} from the data (1.4) and (1.6) as a point $(w_1^1,w_2^1,w_1^2,w_2^2,a_{12},a_{21})$ in the space

$$Y:=C^0(E^1)\times C^0(E^1)\times C^0(E^2)\times C^0(E^2)\times C^0[0,H]\times C^0[0,H],$$

where E^1 and E^2 are the closed triangles in the (x,t)-plane with vertices

$$\begin{aligned} &E^1:\ (0,0),\ (0,2H/c),\ (H,H/c);\\ &E^2:\ (H,0),\ (H,2H/c),\ (0,H/c). \end{aligned}$$

An outline of the proof is as follows.
Step 1: We put the observed data as

$$h_1^1(t):=I_1^1(H,t),\quad h_2^1(t):=I_2^1(0,t),\quad h_1^2(t):=I_1^2(H,t),\quad h_2^2(t):=I_2^2(0,t).$$

Using (2.3) and (2.7), we find the following relations:

$$\begin{aligned} &\bar{I}_2^2(0,t)=I_2^2(0,t)=h_2^2(t),\\ &w_2^1(0,t)=\bar{I}_2^1(0,t)=I_2^1(0,t)=h_2^1(t),\\ &w_1^2(H,t)=\bar{I}_1^2(H,t)=p_1(H)I_1^2(H,t)=p_1(H)h_1^2(t). \end{aligned}$$

Putting $x=0$ in Eq. (2.7b) with $k=2$, we have

$$h_2^2(t)=p_2(H)\delta(t-H/c)+w_2^2(0,t).$$

Since $w_2^2(0,t)$ is a bounded function, we can identify the number $p_2(H)$ as the coefficient of the singular part $\delta(t-H/c)$ of $h_2^2(t)$.
Step 2: In Eq. (2.8b), we substitute $x=0$ and then take the limit as $t\uparrow 2x/c$ (from below) for each value of $x\in[0,H]$. Thus we obtain

$$\begin{aligned} a_{21}(x)=-2\bigg[&h_2^1\Big(\frac{2x}{c}\Big)\\ &+\int_{x/c}^{2x/c} a_{21}(\xi)w_1^1(\xi,\tau)\Big|_{\xi=-c\tau+2x} d\tau\bigg],\quad 0\le x\le H. \end{aligned} \tag{3.1}$$

Similarly we substitute $x=H$ in (2.9a), take the limit $t\uparrow 2(H-x)/c$, and thus

$$a_{12}(x)=\frac{-2}{p_2(H)}\Big[p_1(H)h_1^2\Big(\frac{2(H-x)}{c}\Big)\\ +\int_{(H-x)/c}^{2(H-x)/c} a_{12}(\xi)w_2^2(\xi,\tau)\Big|_{\xi=c\tau-H+2x}d\tau\Big],\quad 0\le x\le H. \tag{3.2}$$

Step 3: We rewrite Eqs. (2.8), (2.9), (3.1), and (3.2) into an equivalent ones for new unknowns

$$y=(y_1,y_2,y_3,y_4,y_5,y_6):=\big(p_2(H)w_1^1,\ p_2(H)w_2^1,\ w_1^2,\ w_2^2,\ a_{21},\ a_{12}\big)\in Y,$$

namely,

$$y_1(x,t)=-\int_{t_1(x,t)}^t y_5(\xi)y_2(\xi,\tau)\Big|_{\xi=c\tau+x-ct}d\tau,\qquad (x,t)\in E^1, \tag{3.3a}$$

$$y_2(x,t)=-\int_{t_2(x,t)}^t y_6(\xi)y_1(\xi,\tau)\Big|_{\xi=-c\tau+x+ct}d\tau-p_2(H)F_2^1(x,t),\qquad (x,t)\in E^1, \tag{3.3b}$$

$$y_3(x,t)=-\int_{t_1(x,t)}^t y_5(\xi)y_4(\xi,\tau)\Big|_{\xi=c\tau+x-ct}d\tau-F_1^2(x,t),\qquad (x,t)\in E^2, \tag{3.3c}$$

$$y_4(x,t)=-\int_{t_2(x,t)}^t y_6(\xi)y_3(\xi,\tau)\Big|_{\xi=-c\tau+x+ct}d\tau,\qquad (x,t)\in E^2, \tag{3.3d}$$

$$y_5(x)=y_5^0(x)-\frac{2}{p_2(H)}\int_{x/c}^{2x/c} y_5(\xi)y_1(\xi,\tau)\Big|_{\xi=-c\tau+2x}d\tau,\qquad x\in[0,H], \tag{3.3e}$$

$$y_6(x)=y_6^0(x)-\frac{2}{p_2(H)}\int_{(H-x)/c}^{2(H-x)/c} y_6(\xi)y_4(\xi,\tau)\Big|_{\xi=c\tau-H+2x}d\tau,\qquad x\in[0,H], \tag{3.3f}$$

where

$$y_5^0(x):=-2h_2^1\Big(\frac{2x}{c}\Big),\qquad y_6^0(x):=\frac{-2}{p_2(H)^2}h_1^2\Big(\frac{2(H-x)}{c}\Big).$$

We have used $p_1(H)p_2(H)=1$ and the fact that $p_2(H)$ has been known in Step 1. We also remark the following facts:

(1) Each of the mappings

$$C^0[0,H]\ni y_5\mapsto p_2(H)F_2^1\in C^0(E^1),\quad C^0[0,H]\ni y_6\mapsto F_1^2\in C^0(E^2)$$

is a bounded operator with norm $\le p_2(H)/2$.

(2) $p_2(H)<1$.

(3) The functions h_2^1 and h_1^2 are continuous and bounded in modulus by a number that depends only on the admissible set $A(m,M)$.

Eqs. (3.3) define a fixed point equation in the Banach space Y. By using the facts (1)-(3) above, it can be concluded that the right-hand sides of (3.3) together define a nonlinear contraction mapping in a closed neighborhood of the point $(0,0,0,0,y_5^0,y_6^0)$ if H is so small that $0<H<H^*$, where the number H^* depends only on the admissible set $A(m,M)$. Thus we find a unique solution to (3.3) by the contraction mapping principle; in particular a_{12} and a_{21} are identified. Consequently, the functions

$$cq\mu_s=\sqrt{a_{12}a_{21}}, \quad p_1=\sqrt[4]{\frac{a_{12}}{a_{21}}},$$

are identified by (2.6). Hence we obtain

$$q\mu_s=\frac{1}{c}\sqrt{a_{12}a_{21}} \tag{3.4}$$

and

$$\mu_a=\frac{1}{4}\frac{d}{dx}(\log a_{12}-\log a_{21})-\frac{1}{c}\sqrt{a_{12}a_{21}} \tag{3.5}$$

from (2.4a).

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