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Continuous and Discrete Adjoint Approach Based on Lattice Boltzmann Method in Aerodynamic Optimization Part I: Mathematical Derivation of Adjoint Lattice Boltzmann Equations

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Abstract. The significance of flow optimization utilizing the lattice Boltzmann (LB) method becomes obvious regarding its advantages as a novel flow field solution method compared to the other conventional computational fluid dynamics techniques. These unique characteristics of the LB method form the main idea of its application to optimization problems. In this research, for the first time, both continuous and discrete adjoint equations were extracted based on the LB method using a general procedure with low implementation cost. The proposed approach could be performed similarly for any optimization problem with the corresponding cost function and design variables vector. Moreover, this approach was not limited to flow fields and could be employed for steady as well as unsteady flows. Initially, the continuous and discrete adjoint LB equations and the cost function gradient vector were derived mathematically in detail using the continuous and discrete LB equations in space and time, respectively. Meanwhile, new adjoint concepts in lattice space were introduced. Finally, the analytical evaluation of the adjoint distribution functions and the cost function gradients was carried out.

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1 Introduction

Adjoint method is one of the gradient-based techniques in which cost function gradient vector with respect to design variables is calculated indirectly by solving an adjoint equ-

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ation. Although, there is an additional cost arising from solving the adjoint equation, the gradients of cost function can be altogether achieved with respect to each design variable. Consequently, the total cost to obtain these gradients is independent of the number of design variables and amounts to the cost of two flow solution roughly [1]. There are two approaches to develop the adjoint equation: continuous and discrete. Continuous adjoint approach utilizes the differential forms of flow field governing equations and cost function. Variations of the cost function and the flow field equations with respect to the flow field variables and the design variables are combined through the use of Lagrange multipliers, also called adjoint variables. Via gathering the terms associated with the variation of the flow field variables, the adjoint equation and its boundary conditions are reached. The terms associated with the variation of the design variables produce the cost function gradient vector. The flow field equations and the adjoint equation with its boundary conditions must finally be discretized using suitable numerical methods. In this approach, physical significance of the adjoint variables is very clear, since we are dealing with a continuous differential equation that can be solved even analytically in some special cases. But, in the discrete adjoint approach, the adjoint equation is directly extracted from a set of discrete flow field equations and cost function gained from numerical approximation of the equations. Discrete adjoint equation is derived by collecting together all the terms multiplied by the variation of the discrete flow variables. Major disadvantage of this approach is the complexity of the adjoint equation derived from the discrete flow field (Navier-Stokes) equations; so that the complete extraction of all the discrete terms in the adjoint equation and the gradient vector requires a lot of algebraic manipulations. In addition, the viscous flux in the Navier-Stokes (NS) equations further increases the complexity of deriving them in viscous flows. The discrete adjoint equation becomes very complicated when the flow field (NS) equations are discretized with higher order schemes and using flux limiters. Therefore cost of discrete equation derivation from the NS equations is more while the implementation of the continuous adjoint method is very simple. The discrete adjoint method requires more computational efforts in comparison with the continuous one. Another critical issue of interest is the relative accuracy of the calculated gradients by the two approaches. The continuous adjoint approach provides the inexact gradient to the exact cost function. On the other hand, the discrete adjoint approach provides the exact gradient to the inexact cost function [2]. Here, the exact cost function is defined as the continuous form of the cost function, and the inexact cost function as the value computed from the discrete field equations and the boundary conditions. In other words, the continuous gradient is calculated from the discretized continuous adjoint equation, derived from the continuous flow field equations and cost function. Therefore, the continuous gradient is not exactly consistent with the cost function which is evaluated numerically. The advantage of the discrete adjoint method is that the resulting discrete gradient is exactly consistent with the discrete cost function and the method does not suffer from this inconsistency. Therefore resulting gradients from discrete method have more consistency with resulting gradients from finite difference method. In this case, however, the difference between discrete and continuous