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A High-Accuracy Finite Difference Scheme for Solving Reaction-Convection-Diffusion Problems with a Small Diffusivity

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Abstract. This paper is devoted to a new high-accuracy finite difference scheme for solving reaction-convection-diffusion problems with a small diffusivity ε. With a novel treatment for the reaction term, we first derive a difference scheme of accuracy $\mathcal{O}(\varepsilon^2 h+\varepsilon h^2+h^3)$ for the 1-D case. Using the alternating direction technique, we then extend the scheme to the 2-D case on a nine-point stencil. We apply the high-accuracy finite difference scheme to solve the 2-D steady incompressible Navier-Stokes equations in the stream function-vorticity formulation. Numerical examples are given to illustrate the effectiveness of the proposed difference scheme. Comparisons made with some high-order compact difference schemes show that the newly proposed scheme can achieve good accuracy with a better stability.

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Key words: Reaction-convection-diffusion equation, incompressible Navier-Stokes equations, boundary layer, interior layer, finite difference scheme.

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1 Introduction

Let Ω be an open, bounded and convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider the following boundary value problem for the scalar reaction-convection-diffusion equation:

$$\begin{cases} -\varepsilon\Delta u+\boldsymbol{a}\cdot\nabla u+\sigma u=f & \text{in } \Omega, \\ u=g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where u is the physical quantity of interest (e.g., concentration of some chemical substance); $\varepsilon>0$ is the diffusivity, which determines the size of diffusion relative to reaction or convection; $\boldsymbol{a}=(a_1(x,y),a_2(x,y))^\top$ is the given convection field; $\sigma\geq 0$ is the reaction coefficient; f is a given source term and g is the prescribed boundary data.

It was already known that the solution u of problem (1.1) may exhibit localized phenomena such as boundary and interior layers when the diffusivity ε is small enough compared with the size of convection field $\boldsymbol{a}$ or the reaction coefficient σ, i.e., the problem is reaction-convection-dominated. Boundary and interior layers are narrow regions in the domain Ω where the solution u changes rapidly. It is often difficult to resolve numerically the high gradients near the layer regions. However, the presence of layers in the solution is very common in many partial differential equations arising from physical science and engineering applications. Besides, problem (1.1) with a small diffusivity ε serves as a vehicle for the study of more advanced incompressible Navier-Stokes equations at high Reynolds numbers. Therefore, the reaction-convection-dominated case of problem (1.1) has been the focus of intense research for quite some time. Unfortunately, most conventional numerical methods for the reaction-convection-dominated problems are lacking in either stability or accuracy (cf. [14, 17, 23]). For example, when the mesh-Péclet number $Pe_h:=\|\boldsymbol{a}\|_\infty h/(2\varepsilon)$ is large (it will occur if $0<\varepsilon\ll 1$), the second-order central difference scheme performs very poorly since large spurious oscillations exhibit not only near the layer regions but also in the others; similar phenomena occur in the finite element method [2].

In this paper, we will focus on developing high-accuracy finite difference schemes for solving problem (1.1) with a small diffusivity ε. Although the usual upwind difference scheme is a simple and stable approach for solving problem (1.1), it exhibits a lower accuracy of $\mathcal{O}(h+k)$, where h and k denote the grid sizes in the x- and y-directions, respectively. On the other hand, higher-order finite difference approximations (that is, approximations whose errors are proportional to $\mathcal{O}(h^m+k^m)$ with $m>2$) are possible, but they typically require non-compact stencils. A compact stencil utilizes eight grid points directly adjacent to the node about which the differences are taken for the discretization. Thus, the use of non-compact stencils complicates numerical formulations near boundaries, increases matrix bandwidth and hence the computational cost. In [20], Spotz proposed a class of fourth-order finite difference schemes that do not look at extra points, but instead approximate the leading terms in the truncation error to obtain higher accuracy. This is achieved by differentiating the governing equation to find, for example, the third derivative in terms of lower-order derivatives that can then be differenced compactly and included in the finite difference formulation. This increases the accuracy while still maintaining an overall compact stencil. However, a more careful calculation shows that the accuracy of the fourth-order compact difference schemes is indeed $\mathcal{O}((h^4+k^4)/\varepsilon)$ and then the accuracy will be deteriorated when the diffusivity ε is getting small (see also [26]). In practice, the fourth-order compact difference approximations may be over-smoothed when the mesh-Péclet number is greater than one.

In this paper, we will devise a new finite difference scheme of accuracy $\mathcal{O}(\varepsilon^2(h+k)+$