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Multiscale Basis Functions for Singular Perturbation on Adaptively Graded Meshes

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Abstract. We apply the multiscale basis functions for the singularly perturbed reaction-diffusion problem on adaptively graded meshes, which can provide a good balance between the numerical accuracy and computational cost. The multiscale space is built through standard finite element basis functions enriched with multiscale basis functions. The multiscale basis functions have abilities to capture originally perturbed information in the local problem, as a result our method is capable of reducing the boundary layer errors remarkably on graded meshes, where the layer-adapted meshes are generated by a given parameter. Through numerical experiments we demonstrate that the multiscale method can acquire second order convergence in the L^2 norm and first order convergence in the energy norm on graded meshes, which is independent of ε. In contrast with the conventional methods, our method is much more accurate and effective.

AMS subject classifications: 35J25, 65N12, 65N30

Key words: Multiscale basis functions, singular perturbation, boundary layer, adaptively graded meshes.

1 Introduction

Singularly perturbed problems have attracted much attention during the past decades. The perturbed parameters in the partial differential equations arise naturally or artificially. Its main difficulty lies in so-called boundary layer behavior, i.e., the solution varies rapidly in a thin boundary layer with a very small parameter ε. Using the standard finite element method (FEM) or finite difference method (FDM) to solve the problem directly

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is much costly and is not independently uniform-consistent. This motivates us to study efficient numerical methods for the singular perturbation problems (see [11, 13, 16]).

In recent years the numerical solutions of singularly perturbed problems have been intensively studied, and there are two major strategies. One is h refinement (h is mesh size) on layer-adapted meshes, e.g., [2, 12, 17, 19]. The other is p refinement (p is degree of approximating polynomials) or hp refinement (the combinations of h and p), e.g., [6, 18, 20]. Chen and Xu [3] presented a mathematical proof on accuracy and stability of the mesh adaptation for one dimensional singular problem. Shishkin [15] proposed a finite difference scheme on a priori adapted meshes for a singularly perturbed parabolic convection-diffusion model. Roos [14] considered a stabilized finite element method on layer-adapted meshes and applied the recovery techniques to acquire supercloseness results.

In addition, the finite element method can be extended to the multiscale scheme. For that purpose, Hou, Wu and Cai [7, 8] proposed the multiscale finite element method (MsFEM) by solving the local homogenization problem for basis functions, and provided convergence analyses and numerical examples for problems with rapidly oscillating coefficients. Araya and Valentin [1] considered the a posterior error estimates for the reaction-diffusion problem, and obtained consistent energy norm estimate. Efendiev and Hou [5] discussed the applications of MsFEM to two-phase immiscible flow simulation in which limited global information was taken into accounted, and the inverse problem was also discussed. Jiang and Huang [9] numerically investigated the MsFEM with rapidly oscillation coefficients and gave a good choice of boundary condition in the local problem for multiscale basis functions. On coarse uniform meshes Jiang and Sun [10] obtained much accurate results with the contributions of analytic singular basis functions to reduce the boundary layer errors remarkably. Efendiev, Galvis and Gildin [4] applied the spectral multiscale finite element method with the combination of local and global model reduction techniques, and achieved a balanced and optimal result in practical applications.

The new point in this paper is to demonstrate the accuracy and efficiency of multiscale basis functions combined with a modified version of graded meshes for singularly perturbed problems. The multiscale bases can capture the local boundary information on the layer-adaptively graded meshes, and therefore offer the uniform-consistent and predictably convergent solutions. When the conventional methods fail in cases, the proposed MsFEM is shown to obtain the accurate layer behaviors and reduce the computational costs, which may be applied in many realms.

The remaining part of this paper is organized as follows. In Section 2 we introduce the singularly perturbed reaction-diffusion model with small parameter ε and build the adaptively graded meshes for our MsFEM. In Section 3 we construct the enriched multiscale space through multiscale basis functions plus standard finite element basis functions. Numerical experiments are provided in Section 4, which demonstrate the efficiency and superiority of MsFEM on graded meshes for the singularly perturbed problem. And finally concluding remarks are given in Section 5.

2 The singularly perturbed problem and adaptively graded meshes

2.1 The reaction-diffusion model

In this paper, we use the Einstein summation convention for repeated indices. $L^2(\Omega)$ denotes the space of square integrable functions defined on the domain Ω, and $H^k(\Omega)$ denotes Sobolev spaces equipped with norms $\|u\|^2_{k,\Omega}=\int_\Omega \sum_{|\alpha|\le k}|D^\alpha u|^2$, where $D^\alpha u$ is the α-th order derivatives of u. $H^1_0(\Omega)$ consists of functions in $H^1(\Omega)$ that vanish on the domain boundary $\partial\Omega$.

Consider the two dimensional reaction-diffusion equation

$$\begin{cases} Lu:=-\varepsilon\Delta u+\sigma u=f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where ε and σ are positive parameters, and f is a smooth function in $L^2(\Omega)$. We know that a quite small ε would bring boundary layer behavior, which will make the efficient discretization very difficult.

The variational formulation of (2.1) is to find $u\in H^1_0(\Omega)$ such that

$$a(u,v)=(f,v) \quad \text{for any } v\in H^1_0(\Omega), \tag{2.2}$$

where

$$a(u,v)=\int_\Omega\Big(\varepsilon\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\varepsilon\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\sigma uv\Big)d\mathbf{x} \tag{2.3}$$

and

$$(f,v)=\int_\Omega fv d\mathbf{x}. \tag{2.4}$$

The bilinear form $a(\cdot,\cdot)$ is coercive and continuous, $\mathbf{x}$ is the two dimensional space, and u is the exact solution.

2.2 The graded meshes

The Shishkin meshes proposed by Shishkin [15] are adapted to the layers structure. Suppose that the boundary layer is located at neighborhood of $(x,y)=(1,1)$, for example we take $\tau=0.1$ as a transition parameter, so the sub-domains are Shishkin piecewise uniform meshes, see the middle of Fig. 1.

Our graded meshes are a modified version of the Shishkin meshes, and they are highly anisotropic and non-uniform. It may be obtained by properly selecting the mesh generating function [12, 19]. They are constructed for both x and y directions in a tensor product way, and N is the partition number in each direction. The transition width is

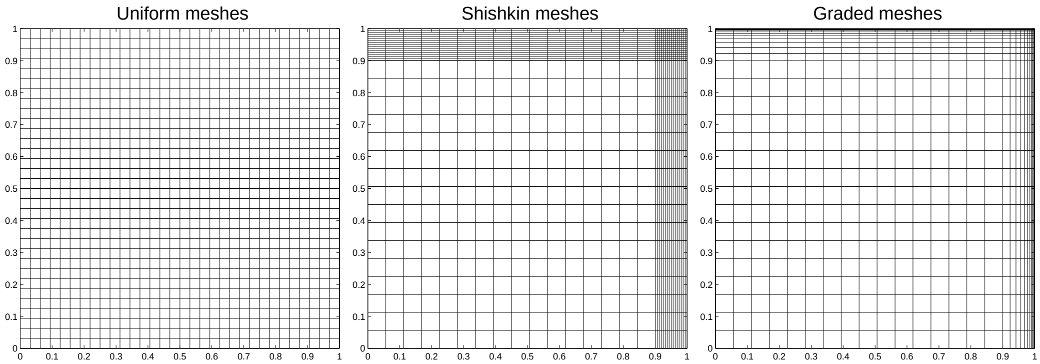

Figure 1: It is shown uniform (U), Shishkin (S) and graded (G) meshes, respectively.

taken as $\tau=\tau_x=\tau_y=\mathcal{O}(\sqrt{\varepsilon})$, and the domain Ω is divided into four sub-domains Ω_k:

$$\Omega_1=[0,1-\tau]^2, \qquad \Omega_2=[1-\tau,1]\times[0,1-\tau], \tag{2.5a}$$

$$\Omega_3=[0,1-\tau]\times[1-\tau,1], \qquad \Omega_4=[1-\tau,1]^2, \tag{2.5b}$$

where $\Omega_k=[x_i,y_j]\in\Omega$: $i,j=1,2,\cdots,N+1$ and x_i, y_j are discrete nodes by

$$x_i=\begin{cases} 2(1-\tau)(i-1)/N, & i=1,\cdots,\frac{1}{2}N+1, \\ 1-\tau(2(N+1-i)/N)^{\lambda}, & i=\frac{1}{2}N+2,\cdots,N+1, \end{cases} \tag{2.6a}$$

$$y_j=\begin{cases} 2(1-\tau)(j-1)/N, & j=1,\cdots,\frac{1}{2}N+1, \\ 1-\tau(2(N+1-j)/N)^{\lambda}, & j=\frac{1}{2}N+2,\cdots,N+1, \end{cases} \tag{2.6b}$$

where the mesh parameter λ is an integer greater than one. An illustration of the graded meshes for the 2D case with $\tau=0.1$, $N=32$ and $\lambda=4$, is shown in the right of Fig. 1.

3 The enriched multiscale method and multiscale basis functions

3.1 The enriched multiscale space

Let $\mathcal{K}_h$ be a partition of Ω into rectangles K with the mesh size h, $0<h\ll 1$. In each element $K\in\mathcal{K}_h$, we define a set of nodal basis $\{\psi_i$ or ϕ_i, $i=1,\cdots,4\}$ at four nodes of rectangular element.

It is well known that the standard finite element space is composed of piecewise bilinear polynomials $\mathbb{Q}^1(K)$,

$$V^h=\{v_h\in H^1(\Omega)|v_{h|_K}\in\mathbb{Q}^1(K),\ \forall K\in\mathcal{K}_h\}, \tag{3.1}$$

and $V_0^h = V^h \cap H_0^1(\Omega)$. The FEM for the reaction-diffusion equation (2.1) is to find $u_g \in V_0^h$ such that

$$a(u_g,v) = (f,v) \quad \text{for any } \ v \in V_0^h. \tag{3.2}$$

In this paper, we construct the multiscale finite element space as

$$U^h = V_0^h \oplus M^h, \tag{3.3}$$

where M^h is spanned by the multiscale basis functions ϕ_i which are solved from a local problem. We assume the multiscale basis are continuous across the boundaries of elements,

$$M^h = \text{span}\{\phi_i : i = 1,\cdots,4, \ \forall K \in \mathcal{K}_h\} \subset H_0^1(\Omega). \tag{3.4}$$

The discrete problem is to seek $u_h \in U^h$ such that

$$a(u_h,v) = (f,v) \quad \text{for any } \ v \in U^h. \tag{3.5}$$

The main difference between MsFEM and FEM is *the construction of basis functions*. We know that in FEM, the standard linear or high order basis functions ψ_i are used to build the discrete space; while MsFEM's main goal is to obtain the macroscopic solutions efficiently by constructing the multiscale basis functions ϕ_i without resolving all microscopic scales. The basis functions ϕ_i are constructed with respect to the differential operator in the local problem. As a consequence they can adaptively reflect the nature of local information such as the singularly perturbed property in our problem.

3.2 Local multiscale basis functions

The multiscale basis functions ϕ_i are constructed in the local problem

$$\begin{cases} L\phi_i := -\varepsilon\Delta\phi_i + \sigma\phi_i = 0 & \text{in } \ K \in \mathcal{K}_h, \\ \phi_i = \theta_i & \text{on } \ \partial K. \end{cases} \tag{3.6}$$

We require $\phi_i(\mathbf{x}_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, i.e., when $i = j$, $\delta_{ij} = 1$, when $i \neq j$, $\delta_{ij} = 0$. Here θ_i is boundary condition and let $\mathbf{x}_j \in \bar{K}$ $(j = 1,\cdots,4)$ be the vertexes of K and are labeled counterclockwise from the lower left corner as (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4) in turn.

To guarantee the well-posedness of local problem (3.6), we define the linear boundary condition to let θ_i vary linearly on ∂K. For example,

$$\theta_1(x,y) = \begin{cases} (x_2 - x)/(x_2 - x_1) & \text{on } \ [x_1,x_2], \\ (y_4 - y)/(y_4 - y_1) & \text{on } \ [y_1,y_4], \end{cases} \tag{3.7}$$

and θ_1 be zero on other two boundaries $[x_4,x_3]$, $[y_2,y_3]$. We can define the other three basis boundary conditions θ_2, θ_3, θ_4 similarly.

With the given boundary condition, we solve the local problem by FEM on subelement size. This would enable the multiscale basis adaptively capture the singular perturbation of differential operator in the boundary domain. As a consequence, small scale property in each element K can be taken into the macroscopic solution through the multiscale basis functions. It should be pointed out that with different ε, the obtained multiscale basis ϕ_i are able to reflect different singular perturbation for the whole problem.

The advantage of multiscale finite element method is its accuracy and computational efficiency. For example, let N be the number of elements and M be the number of subelements in each spatial direction, then there are total $(NM)^d$ elements at fine grid level (here $d=2$ is the dimension). Note that choosing M is a scientific consideration, a wise M may provide perfect balance both for the accuracy and the computational cost. The advanced relationship has not been further investigated in this paper, we just take $M=4$ for simplicity. The computer memory required for the FEM is $\mathcal{O}(N^d M^d)$, while only $\mathcal{O}(N^d+M^d)$ for the MsFEM. Moreover, the multiscale method is naturally adaptive to massive parallel computers, which enables the MsFEM to handle extremely huge system in practice.

4 Numerical results

4.1 Implementation and explanation

In this section, we investigate the accuracy and superiority of the adaptive multiscale finite element method through numerical experiments. For simplicity, the computations are carried out in a unit square domain Ω. We denote N as the partition number in both x and y directions, thus domain Ω is divided into $N\times N$ non-uniform elements with mesh size $h=\max_k h_k$, where h_k is the size of subdomain. To compute the multiscale basis functions in (3.6), boundary elements are divided into $M\times M$ subelements with mesh size $h_s=h_k/M$.

We define the linear boundary condition for multiscale basis functions, and we can verify that $\sum_{i=1}^4 \phi_i \equiv 1$, $\forall K\in\mathcal{K}_h$. After using standard FEM to solve the local problem for basis functions, we compute the gradient of basis at center of subelements, and the local stiffness matrix and local right hand side are computed using two dimensional Gauss numerical quadrature. We glue the local stiffness matrix to the global stiffness matrix and the corresponding global right hand side, then we solve the discrete algebra equations by algebraic multigrid method to acquire the MsFEM solution u_h.

It is known that for the reaction-diffusion problem (2.1) when $\sigma h^2 \ll \varepsilon$, the FEM can give accurate results. However, when ε is very small, i.e., $\varepsilon \ll \sigma h^2$, the FEM Galerkin solution u_g would lead to oscillations and large layer errors. In the following numerical experiments we try to improve the accuracy to some extent by the FEM on our graded meshes. Then we apply the MsFEM to the local problem (3.6) with $h_s^2=(h/M)^2\le\varepsilon$ (we take $M=4$ to construct the localized mesh). We will demonstrate the abilities of multiscale basis functions on graded meshes to capture the boundary information, which reduce the boundary layer errors and obtain high accuracy. Note that the multiscale basis functions are used only near the boundary layers while the standard finite element basis functions are used in the smooth domain, thus we can save the computer memery to $\mathcal{O}_1(N^d)+\mathcal{O}_2(M^d)$ further.

We test the error in the L^2 norm and the energy norm

$$\|u-u_h\|^2_{0,\Omega}=\int_\Omega (u-u_h)^2 d\mathbf{x}, \tag{4.1a}$$

$$\|u-u_h\|^2_{e,\Omega}=\int_\Omega \varepsilon(\nabla(u-u_h))^2+\sigma(u-u_h)^2 d\mathbf{x}, \tag{4.1b}$$

by applying FEM and MsFEM, respectively.

To analyze the efficiency we compare the exact solution with different numerical solutions. In the following tables, L^2 norm and energy norm error in (4.1a) and (4.1b) of discrete grid are listed under the column FEM (G), MsFEM and MsFEM (G), respectively. Here (G) means on the graded meshes, and the convergence order is listed as the mesh refinement.

4.2 Experiment results

In problem (2.1), we set $\Omega=[0,1]\times[0,1]$, $\sigma=1$ and exact solution

$$u=xy(1-e^{\frac{x-1}{\varepsilon}})(1-e^{\frac{y-1}{\varepsilon}}),$$

and corresponding right hand

$$f=\Big(2+\frac{x}{\varepsilon}\Big)y(1-e^{\frac{y-1}{\varepsilon}})e^{\frac{x-1}{\varepsilon}}+\Big(2+\frac{y}{\varepsilon}\Big)x(1-e^{\frac{x-1}{\varepsilon}})e^{\frac{y-1}{\varepsilon}}+xy(1-e^{\frac{x-1}{\varepsilon}})(1-e^{\frac{y-1}{\varepsilon}}).$$

Clearly the boundary layer is located at neighborhood of $x=1$ and $y=1$, and we take the transition width $\tau=\sqrt{\varepsilon}$ and $10\sqrt{\varepsilon}$ for two parameters ε.

From Table 1 we observe when $\varepsilon=1e-2$ is not too small, three numerical methods are effective for the model, and they can obtain second order convergence rate (i.e., CN^{-2}) in the L^2 norm and first order convergence rate (i.e., CN^{-1}) in the energy norm with the mesh refinement. However, we note that the accuracy of MsFEM (G) is the best. In Fig. 2, there are no obvious boundary layer phenomena, and it is evident the MsFEM (G) can solve it more precisely.

From Table 2, when $\varepsilon=1e-6$ is very small, we find that the numerical result of MsFEM (G) is one-order of magnitude higher than that of FEM (G), and it is much superior to MsFEM's. There are second order L^2 norm and first order energy norm convergence rate with the mesh refinement on graded meshes. The boundary layers near $x=1$ and $y=1$ are shown in Fig. 3, but the error of MsFEM (G) is much more accurate than those of FEM (G) and MsFEM, and this is owing to the contributions of multiscale basis functions to resolve the singular perturbation in boundary layers.

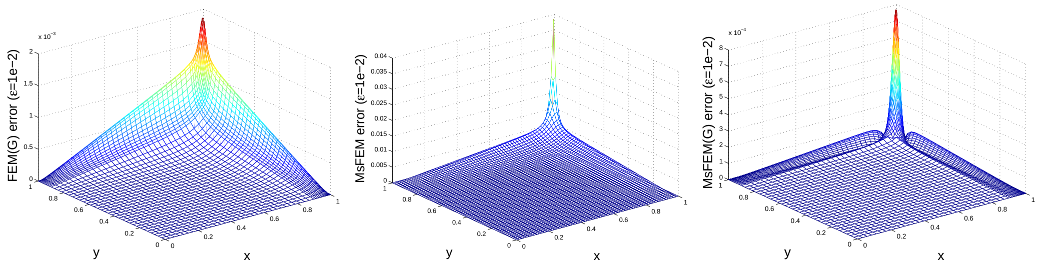

Figure 2: Errors of FEM (G), MsFEM and MsFEM (G) with $\varepsilon = 1.0e-2$ and $N = 64$.

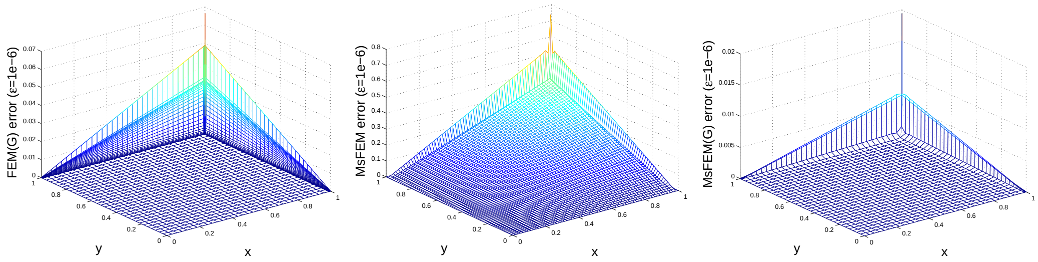

Figure 3: Errors of FEM (G), MsFEM and MsFEM (G) with $\varepsilon = 1.0e-6$ and $N = 64$.

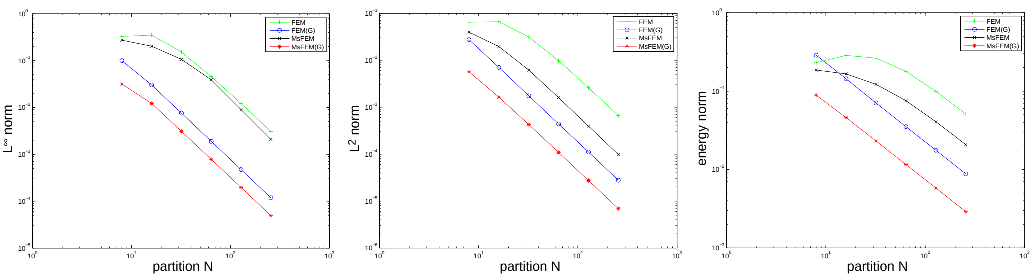

Figure 4: Norm results in the log-log scale with $\varepsilon = 1.0e-2$ by FEM, FEM (G), MsFEM and MsFEM (G), respectively.

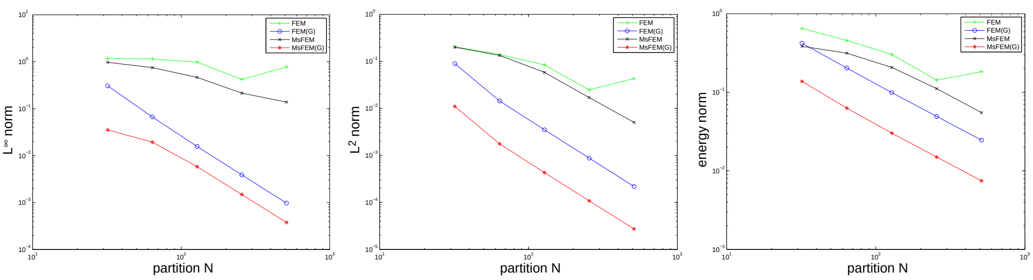

Figure 5: Norm results in the log-log scale with $\varepsilon = 1.0e-6$ by FEM, FEM (G), MsFEM and MsFEM (G), respectively.

In Fig. 4 and Fig. 5, we present three norm errors of L^∞, L^2 and energy with $\varepsilon = 1e-2$ and $1e-6$. It is apparent that FEM on uniform meshes gives bad accuracy and divergence

Table 1: Errors and convergence rates by FEM (G), MsFEM and MsFEM (G) with $\varepsilon=1.0e-2$.

N	L^2 norm					
	FEM (G)	order	MsFEM	order	MsFEM (G)	order
8	2.739e-2		3.957e-2		5.659e-3	
16	7.021e-3	1.96	1.954e-2	1.02	1.633e-3	1.79
32	1.747e-3	2.01	6.172e-3	1.66	4.262e-4	1.94
64	4.400e-4	1.99	1.596e-3	1.95	1.086e-4	1.97
128	1.107e-4	1.99	3.944e-4	2.02	2.738e-5	1.99
256	2.776e-5	2.00	9.802e-5	2.01	6.874e-6	1.99
N	energy norm					
	FEM (G)	order	MsFEM	order	MsFEM (G)	order
8	2.874e-1		1.858e-1		8.879e-2	
16	1.438e-1	1.00	1.660e-1	0.16	4.596e-2	0.95
32	7.060e-2	1.03	1.221e-1	0.44	2.315e-2	0.99
64	3.523e-2	1.00	7.565e-2	0.69	1.159e-2	1.00
128	1.760e-2	1.00	4.073e-2	0.89	5.799e-3	1.00
256	8.799e-3	1.00	2.079e-2	0.97	2.900e-3	1.00

Table 2: Errors and convergence rates by FEM (G), MsFEM and MsFEM (G) with $\varepsilon=1.0e-6$.

N	L^2 norm					
	FEM (G)	order	MsFEM	order	MsFEM (G)	order
32	8.930e-2		1.993e-1		1.098e-2	
64	1.435e-2	2.64	1.334e-1	0.58	1.759e-3	2.64
128	3.497e-3	2.04	5.826e-2	1.20	4.306e-4	2.03
256	8.708e-4	2.01	1.694e-2	1.78	1.079e-4	2.00
512	2.177e-4	2.00	5.027e-3	1.75	2.722e-5	1.99
N	energy norm					
	FEM (G)	order	MsFEM	order	MsFEM (G)	order
32	4.219e-1		3.850e-1		1.380e-1	
64	2.036e-1	1.05	3.148e-1	0.29	6.284e-2	1.13
128	9.922e-2	1.04	2.072e-1	0.60	3.023e-2	1.06
256	4.933e-2	1.01	1.113e-1	0.90	1.498e-2	1.01
512	2.460e-2	1.00	5.508e-2	1.01	7.463e-3	1.01

in the case of small $1e-6$. On the contrary FEM on graded meshes improves the accuracy to some extent, and MsFEM on graded meshes can provide the most accurate simulation and convergence. The superiority of MsFEM (G) for reaction-diffusion boundary layer is obvious.

5 Conclusions

In this paper, an enriched multiscale finite element method is proposed to solve the two dimensional singularly perturbed reaction-diffusion problem, and the multiscale basis functions are combined with a modified version of graded meshes for accuracy and efficiency. The multiscale basis functions are capable of capturing the local singular perturbation to recover the boundary layer errors on the exponentially graded meshes, and therefore offer the numerically stable and uniform-consistent results with the parameter ε. This method is more accurate compared with the conventional methods, and it provides second order convergence in the L^2 norm and first order convergence in the energy norm. The method is not restricted to the reaction-diffusion model, and it can be extended to convection-diffusion problems.

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